phys522: HW #11

1. The Hamiltonian for a charged particle in the presence of EM fields is

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

Show that the Schrodinger equation is invariant under the gauge transformation ($\Lambda(\vec{r},t)$ an arbitrary function)

$$\psi \to \psi' = e^{iq\Lambda/(\hbar c)}\psi$$

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

$$\vec{A} \to \vec{A'} = \vec{A} + \vec{\nabla}\Lambda$$

2. In this problem you calculate the Casimir energy between two parallel conducting plates with area A and separation d in the x-direction (x direction is perpendicular to the plates). The Casmir energy Λ_C is the difference between the vacuum energy between the plates, minus what the energy would be in the same volume Ad without the plates,

$$\Lambda_{\mathrm{C}} = \Lambda_{\mathrm{plates}} - \Lambda_{\mathrm{no-plates}}$$

Start by calculating $\Lambda_{no-plates}$ by summing the zero point energies $2\sum_{\vec{k}}\frac{1}{2}\hbar\omega$ (factor 2 in front is for 2 polarizations) with continuous $\vec{k}$. Separate the modes parallel and perpendicular (x) to the plates as $\vec{k}_{\|} = k_y\hat{y} + k_z\hat{z}$ and $k^2 = \left|\vec{k}_{\|}\right|^2 + k_x^2$

Show that $\Lambda_{no-plates} = A\hbar c\int_0^\infty dx F(x)$ where (with $y = \left|\vec{k}_{\|}\right|$)

$$F(x) = \frac{1}{2\pi}\int_0^\infty y\sqrt{y^2 + \left(\frac{\pi x}{d}\right)^2}\,dy$$

The $\Lambda_{\mathrm{plates}}$ is the same with x replaced by an integer n and the integral over F replaced by a sum over $n = 0$ to ∞. One tricky part is that for $n = 0$ there is only 1 polarization. Why? Another tricky part is that the integrals need to be cut off so as not to diverge at the upper limit ∞. The physical reason for this is that at some very high frequency the plates must become transparent. This translates to $F(\infty) = 0$ as well as all derivatives evaluated at ∞. (Hint for F''' integral, with $z = \left(\frac{\pi x}{d}\right)^2$, make the substitution $s = y^2 + z^2$.)

Use the Euler-Maclaurin formula,

$$
\begin{aligned}
\sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dx\, F(x) &= -\frac{1}{2}\left[F(0) + F(\infty)\right] + \frac{1}{12}\left[F'(\infty) - F'(0)\right] \\
&- \frac{1}{720}\left[F'''(\infty) - F'''(0)\right] + \ldots
\end{aligned}
$$

Figure 1: Euler-Maclaurin formula where prime denotes differentiation with respect to x