phys522: HW #12

1. Using the commutation relations for α^i and β prove the identity for the γ matrices $\gamma^\mu = (\beta, \beta\vec{\alpha})$
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$
where $\mathrm{g}^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$

2. Show that each wave function component of the Dirac equation satisfies the Klein-Gordon equation.
$$\left(\Box + m^2\right)\psi_i = 0$$
where $\Box = \partial^\mu\partial_\mu$ is the 4 dimensional Laplacian. The Dirac equation (in units in which $\mathrm{c} = \hbar = 1$)
$$(\mathrm{i}\gamma^\mu\partial_\mu - \mathrm{m})\psi = 0$$
where ψ is the 4-component spinor wave function.

3. Prove that that helicity is conserved, spin $\frac{1}{2}\vec{\Sigma}$ is not conserved, but total $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ is conserved.
$$\vec{\Sigma} = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

4. The electron Dirac spinor ψ satisfies
$$\left[\gamma^\mu\left(i\partial_\mu + eA_\mu\right) - m\right]\psi = 0$$
Take the charge conjugation operator to be $C = i\gamma^2\gamma^0$. The charge-conjugate spinor $\psi_c = C\gamma^0\psi^*$ can be shown to satisfy
$$\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi_c = 0$$
Show that the charge conjugate current
$$\mathrm{j}^\mu_\mathrm{c} = -\mathrm{e}\bar{\psi}_\mathrm{c}\gamma^\mu\psi_\mathrm{c} = +\mathrm{e}\bar{\psi}\gamma^\mu\psi$$
when you insert an extra minus sign by hand. This shows that the charge conjugate current is the positron current. The minus sign is related to the connection between spin and statistics. In field theory, it occurs because of the antisymmetric nature of the fermion fields.

   Hint: First show the $C^{-1}\gamma^\mu C = (-\gamma^\mu)^T$ and that $\bar{\psi}_C = -\psi^T C^{-1}$.