phys522: HW #2

1. The probability density is for the hydrogen atom $\rho_{n\ell m} = |\psi_{n\ell m}|^2$ and the only part of the wave function that is imaginary comes from the ϕ dependence. We can therefore write the wave function as:

$$\psi_{n\ell m}(\vec{r}) = |\psi_{n\ell m}(\vec{r})|e^{im\phi} = \sqrt{\rho_{n\ell m}(\vec{r})}e^{im\phi}$$

   Show that the probability current is proportional to the quantum number m and given by

$$\vec{J}_{n\ell m} = m\frac{\hbar}{\mu}\frac{\rho_{n\ell m}(\vec{r})}{r\sin\theta}\hat{e}_\phi$$

   Comment on the phrase "stationary states".

2. Commins 8.6

8.6. (a) Consider a bound-state wave function $\psi(\boldsymbol{r})$ with corresponding probability density $\rho = \psi^*\psi$ and probability current density

$$\boldsymbol{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

Assume that ρ satisfies the condition

$$\int \rho\boldsymbol{r}\times\hat{\boldsymbol{n}}\,ds = 0 \tag{1}$$

where the quantity on the left-hand side of (1) is a surface integral taken over a surface at ∞, and $\hat{\boldsymbol{n}}$ is a unit normal vector to the surface. Show that the expectation value of the orbital angular-momentum operator $\boldsymbol{L}$ is

$$\langle \boldsymbol{L}\rangle = m\int \boldsymbol{r}\times\boldsymbol{j}\,d\tau \tag{2}$$

where the integral on the right-hand side of (2) is a volume integral taken over all space.

(b) Now, considering the description of a particle with spin-½ in nonrelativistic quantum mechnics, let us assume that $\psi = \phi(\boldsymbol{r})\chi$, where ϕ is a bounded spatial wave function and χ is a two-component spinor. We write the expectation value of the spin operator as

$$\langle \boldsymbol{S} \rangle = \frac{\hbar}{2}\int \psi^* \boldsymbol{\sigma} \psi \quad d\tau \tag{3}$$

Show that

$$\psi^* \boldsymbol{\sigma} \psi = \frac{1}{2}\boldsymbol{r} \times \left[\nabla \times (\psi^* \boldsymbol{\sigma} \psi)\right] - \frac{1}{2}\nabla\left[\boldsymbol{r} \cdot \psi^* \boldsymbol{\sigma} \psi\right] + \frac{1}{2}\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[x_i \left(\psi^* \boldsymbol{\sigma} \psi\right)\right] \tag{4}$$

and show that the last two terms on the right-hand side of (4) make no contribution to the integral in (3). Thus, in analogy to (2), (3) can be written as

$$\langle \boldsymbol{S} \rangle = m\int \boldsymbol{r} \times \boldsymbol{j}_S \ d\tau \tag{5}$$

Here $\boldsymbol{j}_S$, the *spin current density*, is $\boldsymbol{j}_S = \nabla \times \boldsymbol{V}_S$, where

$$\boldsymbol{V}_S = \frac{\hbar}{4m}\psi^* \boldsymbol{\sigma} \psi \tag{6}$$

(c) Calculate the spin current density in atomic units for the $2^2s_{1/2}$, $m_s = 1/2$ state of atomic hydrogen.