Phys 522

Dirac Equation

See Halzen & Martin, Quarks & Leptons

Use natural units, $\hbar=1$, $c=1$.
Relativistic energy momentum.

$$E^2 = p^2 + m^2$$

Four vector notation:

$$x^\mu = (t, \vec{x}) \qquad x_0 = t$$

upper index, contravariant (transforms oposite basis)

with metric $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$
Repeated indicies summed (Einstein convention)

$$x^\mu x^\nu g_{\mu\nu} = x^\mu x_\nu = t^2 - |\vec{x}|^2$$

derivation $\quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x_0}, \vec{\nabla}\right)$ lower index, covariant

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, -\vec{\nabla}\right)$$

$$\partial^\mu \partial_\mu = \frac{\partial^2}{\partial x_0^2} - \nabla^2 \equiv \Box \quad \text{4 dimentional Laplacian}$$

Klein-Gordon equation for scalar particle like pion.

$$\vec{p} \to -i\vec{\nabla} \qquad E \to i\frac{\partial}{\partial t}$$

$$-\frac{\partial^2}{\partial t^2}\phi = -\nabla^2\phi + m^2\phi$$

$$\Box\phi + m^2\phi = 0$$

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probability flux 4-vector

$$j^\mu = (\rho, \vec{j}) = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

Conserved, covariant current

$$\partial_\mu j^\mu = 0$$

ρ is timelike component of 4-vector.
under Lorentz transformation, $\gamma = (1-v^2)^{-1/2}$

$$\rho' = \gamma \rho \qquad d^3x' = \frac{1}{\gamma} d^3x$$ Lorentz contracted volume

so $\rho' d^3x' = \rho d^3x$

free particle solution

$$\phi = N e^{-ip \cdot x}$$ $p^\mu = (E, \vec{p})$, $p \cdot x = p^\mu x_\mu$

energy eigenvalues

$$E = \pm (p^2 + m^2)^{1/2}$$

$E < 0$ with $\rho < 0$ negative energy solutions

Pauli, Weisskopf put in charge

$$j^\mu = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

ρ is then charge density
$E < 0$ solution $\Rightarrow$ $E > 0$ solution for +charge particle

Dirac – equation first order in time, like Schrodinger.

$$i \frac{\partial}{\partial t} \psi = H \psi$$

$$H = \vec{\alpha} \cdot \vec{p} + \beta m \quad \text{with}$$

$$H^2 = p^2 + m^2 = (\vec{\alpha} \cdot \vec{p} + \beta m)^2 \quad \text{requires}$$

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij} \quad \underline{\text{anti-commutators}}$$

$$\{\alpha^i, \beta\} = 0 \ , \quad \beta^2 = 1$$

Since H is Hermitian, $\vec{\alpha}^\dagger = \vec{\alpha}$, $\beta^\dagger = \beta$

$$\alpha^{i2}, \beta^2 = 1 \quad \Rightarrow \pm 1 \text{ eigenvalues}$$

$$(\alpha^i \beta + \beta \alpha^i) \beta = 0$$

$$\alpha^i = -\beta \alpha^i \beta$$

Cyclic property of trace

$$tr(\alpha^i) = -tr(\beta \alpha^i \beta) = -tr(\alpha^i \beta^2)$$

$$tr(\alpha^i) = -tr(\alpha^i) = 0$$

__traceless__ eigenvalues implies dimension __even__
we know for dim = 2, only 4 linearly independent matrices are pauli + identity

$$\sigma^i, I_{2\times 2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Smallest dimension to satisfy algebra is 4.

The Pauli-Dirac representation diagonalizes

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad I \ 2\times 2$$

then $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

Covariant matrices are $\gamma^\mu = (\beta, \beta\vec{\alpha})$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

The Weyl representation separates left-right "chiral" components

Weyl: $\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

Dirac equation for 4-component spinor ψ,

$$i\frac{\partial}{\partial t}\psi = \left(-i\,\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

Covariant form, multiply on left w/ β

$$i\frac{\partial}{\partial t}\gamma^0\psi + i\,\vec{\gamma}\cdot\vec{\nabla}\psi - \beta^2 m\psi = 0$$

$$\boxed{i\gamma^\mu\partial_\mu\psi - m\psi = 0}$$

recall $\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

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<u>Hermitian conjugate</u> $(\vec{\sigma}^\dagger = \vec{\sigma})$

$$\gamma^{0\dagger} = \gamma^0 \qquad \vec{\gamma}^\dagger = -\vec{\gamma}$$

$$\left(i\gamma^0 \partial_0 \psi + i\vec{\gamma}\cdot\vec{\nabla}\psi - m\psi = 0 \right)^\dagger$$

$$-i\,\partial_0 \psi^\dagger \gamma^0 + i\,\vec{\nabla}\psi^\dagger \cdot \vec{\gamma} - m\psi^\dagger = 0$$

this equation is not Lorentz invariant.
multiply on right with γ^0, use

$$\gamma^{0\,2} = 1 \;, \quad \gamma^0 \vec{\gamma} = -\vec{\gamma}\gamma^0$$

define $\bar{\psi} \equiv \psi^\dagger \gamma^0$

$$-i\,\partial_0 \bar{\psi}\gamma^0 - i(\vec{\nabla}\bar{\psi})\cdot\vec{\gamma} - m\bar{\psi} = 0$$

or manifestly covariant,

$$\boxed{i\,\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0}$$

$\bar{\psi}$ is adjoint (row) spinor

<u>Conserved current:</u> Add

$$\bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$$

$$\left(i\,\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi}\right)\psi = 0$$

$$i\,\partial_\mu \bar{\psi}\gamma^\mu + i\,\bar{\psi}\gamma^\mu\partial_\mu = i\,\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) = 0$$

Conserved current: $j^\mu = -e\,\bar{\psi}\gamma^\mu\psi$

plane wave solution

$$\psi = u(\vec{p})\, e^{-ip\cdot x} \qquad \text{when } p\cdot x = Et - \vec{p}\cdot\vec{x} = 0$$

note - u does not depend on x

convenient Feynman slash notation

$$\not{A} \equiv \gamma^\mu A_\mu$$

direct substitution of ψ into Dirac eq.

$$(\gamma^0 E + i\vec{\gamma}\cdot\vec{\nabla} - m)\, u = 0 \qquad \vec{p} = -i\vec{\nabla}$$

$$p_\mu = (E, -\vec{p}) \qquad \gamma^\mu p_\mu = \gamma^0 E - \vec{\gamma}\cdot\vec{p}$$

u equation: $(\gamma^\mu p_\mu - m)\, u(\vec{p}) = 0$

$$(\not{p} - m)\, u(\vec{p}) = 0$$

where $E^2 = p^2 + m^2$

u equation: $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ up, down 2-spinors

$$H u = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E u$$

when 2nd row has been multiplied by -1

define spinor basis with respect to some z-direction

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\sigma_z}{2}\chi^s = (-1)^{s+1}\tfrac{1}{2}\chi^s \qquad \text{eigenstates of } \hat{S}_z = \frac{\sigma_z}{2}$$

spinor index $s = 1, 2$

Coupled eigenvalue equation

$$\vec{\sigma}\cdot\vec{p}\, u_B = (E-m)\, u_A$$

$$\vec{\sigma}\cdot\vec{p}\, u_A = (E+m)\, u_B$$

for $E>0$, we can take two solution as $u_A^s = \chi^s$. Then

$$u_B^s = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^s$$

giving $s=1,2$
$$u^s = N\begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^s \end{pmatrix} \quad E>0$$

for $E<0$ take $u_B^s = \chi^s$ and get

$s+2 = 3,4$
$$u^{s+2} = N\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^s \\ \chi^s \end{pmatrix} \quad E<0$$

$s=3$ χ^1
$s=4$ χ^2

2-fold degeneracy means there is another operator that commutes with H, the helicity operator

$$\vec{\Sigma}\cdot\hat{p} = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} \qquad \hat{p} = \frac{\vec{p}}{|\vec{p}|} \text{ unit}$$

Spin along direction of motion (helicity) is conserved.

<u>Normalization</u> taken to be

$$\frac{1}{V}\int \psi^\dagger \psi \, d^3x = U^\dagger U = 2E$$

$$(U^s)^\dagger U^s = |N|^2 \left[1 + \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^2\right] = 2E$$

$$|N|^2 \left[1 + \frac{p^2}{(E+m)^2}\right] = 2E$$

$$\frac{|N|^2}{(E+m)^2}\left[(E+m)^2 + p^2\right] = 2E$$

$$E^2 + m^2 + 2Em + p^2 = 2E(E+m)$$

gives $|N|^2 = E+m$ $\quad N = \sqrt{E+m}$

<u>anti-particle</u>: interpretation of <u>negative</u> energy solutions

$$U^{s+2}(\vec{p}) = \sqrt{|E|+m}\begin{pmatrix} \dfrac{-\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi^s \\ \chi^s \end{pmatrix}$$

$e^+ \xrightarrow{\Rightarrow} \quad \longrightarrow \quad \xleftarrow{\Leftarrow} \quad e^-$ $E<0$

$E>0$ $\quad \vec{p}, \vec{\sigma} \qquad\qquad -\vec{p}, -\vec{\sigma}$

<u>define</u> $p^0 = -E > 0$

$$U(-\vec{p}, s)\, e^{-i(Et+\vec{p}\cdot\vec{x})}$$

$$\equiv V(\vec{p}, s')\, e^{i(p_0 t - \vec{p}\cdot\vec{x})} = V(\vec{p}, s')\, e^{ip\cdot x}$$

$s = 3,4 \rightarrow s' = 2,1$ spin flip

$$V(\vec{p}, 2) = \sqrt{p_0+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi^1 \\ \chi^1 \end{pmatrix}$$

$U(\vec{p})$ satisfies Dirac

$$(i\gamma^\mu \partial_\mu - m)\, u(\vec{p})\, e^{-ip\cdot x} = 0$$

$$(i\gamma^\mu p_\mu - m)\, U(\vec{p}) = 0$$

then $\vec{p} \rightarrow -\vec{p}$ $\quad -E = p^0 > 0$

$$(i\gamma^\mu p_\mu + m)\, V(\vec{p}) = 0 \qquad p = (p_0, \vec{p})$$

Charge conjugation operator

Em interaction $q = -e$ $\quad \partial^\mu = (\frac{\partial}{\partial t}, -\vec{\nabla})$

$$p^\mu \to p^\mu + eA^\mu \quad ; \quad \partial^\mu \to i\partial^\mu + eA^\mu$$

then Dirac

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\,\psi = 0$$

c.c. $\quad [\gamma^{\mu *}(-i\partial_\mu + eA_\mu) - m]\,\psi^* = 0$

if charge conjugation matrix can be found such that

$$-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0) \quad \text{then}$$

$$[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\, C\gamma^0\psi^* = 0$$

$$\psi_c = C\gamma^0\psi^* = C\bar{\psi}^T$$

$$C = i\gamma^2\gamma^0 \qquad C\gamma^0 = i\gamma^2 = \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{pmatrix} \quad \text{off diagonal}$$

Spinor solution 1 :

$$\psi_c^{(1)} = i\gamma^2 \left(u^1(\vec{p})\, e^{-ip\cdot x}\right)^* = \sqrt{E+m}\; i\gamma^2 \begin{pmatrix} 1 \\ 0 \\ \vec{\sigma}\cdot\vec{p}/(E+m) \\ 0 \end{pmatrix}$$

$$= \sqrt{E+m} \begin{pmatrix} 0 \\ -\vec{\sigma}\cdot\vec{p}/(E+m) \\ 0 \\ 1 \end{pmatrix} e^{ip\cdot x}$$

$$= u^4(-\vec{p})\, e^{ip\cdot x} = v^1(\vec{p})\, e^{ip\cdot x} \quad \text{with } p_0 = |E|$$

~~spinor~~ positron solution 1

Large + Small components

$$\vec{\sigma}\cdot\vec{p}\, U_B = (E-m)\, U_A$$
$$\vec{\sigma}\cdot\vec{p}\, U_A = (E+m)\, U_B$$

N.R. limit : $E = m\sqrt{\frac{p^2}{m^2}+1} = m + \frac{1}{2}\frac{p^2}{m}$

take $\vec{p} = p\hat{z}$ use $U_A = \chi^s$
$$\sigma_z \chi^s = (-1)^{s+1}\chi^s$$

$$\vec{\sigma}\cdot\vec{p}\, U_A = p(-1)^{s+1}\chi^s = \Big(\underbrace{2m + \tfrac{1}{2}\tfrac{p^2}{m}}_{\doteq 2m}\Big) U_B$$

then in N.R. limit

$$U_B = \frac{p}{2m}(-1)^{s+1}\chi^s = \frac{p}{2m}(-1)^{s+1} U_A$$
$$= \frac{1}{2} v\, U_A$$

putting back speed of light

$$|U_B| = \frac{1}{2}\left(\frac{v}{c}\right)|U_A|$$

So for the E>0 (electro) solutions, the lower 2 components are small in the non-relativistic limit. Similarly, for the positron solutions, the upper two components are small in the nonrelativistic limit.

Non-relativistic Limit

to include Em interaction, $p^\mu \to p^\mu + eA^\mu$
$(q = -e)$

define $E' = E + eA^0$

$$\vec{p}\,' = \vec{p} + e\vec{A} = -i\vec{\nabla} + e\vec{A}$$

To use $\vec{\nabla}$ need spatial part of wave function;

$$\psi = U(\vec{p})\, e^{-i p \cdot x} \qquad U = U_A, U_B$$

Coupled equations are

$$\vec{\sigma}\cdot\vec{p}\,'\, \psi_B = (E' - m)\, \psi_A$$

$$\vec{\sigma}\cdot\vec{p}\,'\, \psi_A = (E' + m)\, \psi_B$$

then

$$(\vec{\sigma}\cdot\vec{p}\,')^2 \psi_A = (\vec{\sigma}\cdot\vec{p}\,')(E' + m)\, \psi_B$$

reasonable to assume $|eA^0| \ll m$, then
$(E' + m) \approx (E + m)$ and

$$(\vec{\sigma}\cdot\vec{p}\,')^2 \psi_A = (E + m)\, \vec{\sigma}\cdot\vec{p}\,' \psi_B = (E + m)(E' - m)\, \psi_A$$

$$(\vec{\sigma}\cdot\vec{p}\,')^2 = p'^2 + i\vec{\sigma}\cdot(\vec{p}\,' \times \vec{p}\,') = p'^2 + ie\vec{\sigma}\cdot(\vec{p}\times\vec{A} + \vec{A}\times\vec{p})$$

$$(\vec{p}\times\vec{A} + \vec{A}\times\vec{p})\, \psi_A = -i\,(\vec{\nabla}\times\vec{A} + \vec{A}\times\vec{\nabla})\, \psi_A$$

$$= -i\,\left(\vec{\nabla}\psi_A \times \vec{A} + (\nabla\times\vec{A})\, \psi_A + \vec{A}\times\vec{\nabla}\psi_A \right)$$

$$= -i\,(\vec{\nabla}\times\vec{A})\, \psi_A = -i\,\vec{B}\, \psi_A$$

Then,

$$(p'^2 + e\vec{\sigma}\cdot\vec{B})\psi_A = (E+m)(E'-m)\psi_A$$

define $E = E_{NR} + m$

and $(E+m)(E'-m) = (E_{NR}+2m)(E_{NR}+eA^0)$

$$\approx 2m(E_{NR}+eA^0)$$

giving

$$\frac{1}{2m}p'^2\psi_A + \frac{e}{2m}\vec{\sigma}\cdot\vec{B}\,\psi_A - eA^0\psi_A = E_{NR}\psi_A$$

$$\boxed{\frac{1}{2m}\left(\vec{P}+e\vec{A}\right)^2\psi_A + \frac{e}{2m}\vec{\sigma}\cdot\vec{B}\,\psi_A - eA^0\psi_A = E_{NR}\psi_A}$$

term $\frac{e}{2m}\vec{\sigma}\cdot\vec{B} = \frac{e}{m}\vec{S}\cdot\vec{B}$

putting in factors $\hbar, c$ and $\mu_B = \frac{e\hbar}{2mc}$ becomes

$$2\mu_B\vec{S}\cdot\vec{B} \quad \text{so } \boxed{g=2}$$