Phys 522

Identical Particles

following Sakurai

Quantum identical partiles (e.g. electrons) are in principle indistinguishable. The Quantum state cannot contain more information than is possible to measure in principle.

Direct Product two particle state

$$|a\rangle|b\rangle$$

a, b complete set of quantum numbers specifying state. Order is understood as particle label 1, 2.

Any linear combination is a valid wave function,

$$|\psi\rangle = c_1 |a\rangle|b\rangle + c_2|b\rangle|a\rangle$$

Introduce operators, $A_1 |a\rangle|b\rangle = a|a\rangle|b\rangle$
$A_2|a\rangle|b\rangle = b|a\rangle|b\rangle$

then $A_1 A_2 |\psi\rangle = ab|\psi\rangle$ quantum numbers a, b do not uniquely specify state. This is called exchange degeneracy.

Particle exchange operator,

$$P_{12}|a\rangle|b\rangle = |b\rangle|a\rangle$$

$\hat{P}_{12} = \hat{P}_{21}$ and $(\hat{P}_{12})^2 = I$.

eigenstates of $\hat{P}_{12}$ have two possible eigenvalues

define $|\psi_\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|a\rangle|b\rangle \pm |b\rangle|a\rangle\right)$

$$\hat{P}_{12}|\psi_\pm\rangle = \pm|\psi_\pm\rangle$$

Requiring $|\psi\rangle$ to be eigenstate of $\hat{P}_{12}$ lifts exchange degeneracy.

Transformation of operators

$$A_1|a\rangle|b\rangle = a|a\rangle|b\rangle$$

$$\hat{P}_{12}A_1|a\rangle|b\rangle = a|b\rangle|a\rangle$$

$$P_{12}^{-1}P_{12} \quad \text{get}$$

$$\left(P_{12}A_1P_{12}^{-1}\right)P_{12}|a\rangle|b\rangle = \left(P_{12}A_1P_{12}^{-1}\right)|b\rangle|a\rangle = a|b\rangle|a\rangle$$

we see $P_{12}A_1P_{12}^{-1} = A_2$.

suppose $[H, \hat{P}_{12}] = 0$ then $|\psi\rangle$ will have simultaneous eigenvalue so energy eigenstates will be $|\psi_\pm\rangle$

We can define symmetrization, antisymmetrization operators

$$\hat{S}_{12} \equiv \frac{1}{2}(1 + P_{12}) \; ; \; \hat{A}_{12} \equiv \frac{1}{2}(1 - P_{12})$$

Then for example, arbitrary linear combination,

$$|\psi\rangle = c_1 |a\rangle|b\rangle + c_2 |b\rangle|a\rangle$$

$$\hat{S}_{12}|\psi\rangle = \frac{1}{2}\Big(c_1 |a\rangle|b\rangle + c_2|b\rangle|a\rangle + c_1 |b\rangle|a\rangle + c_2 |a\rangle|b\rangle\Big)$$

$$= \frac{1}{2}(c_1+c_2)\Big(|a\rangle|b\rangle + |b\rangle|a\rangle\Big) \quad \text{symmetrized}$$

Obvious extentention to n identical particles

Spin-Statistics Theorem of QFT

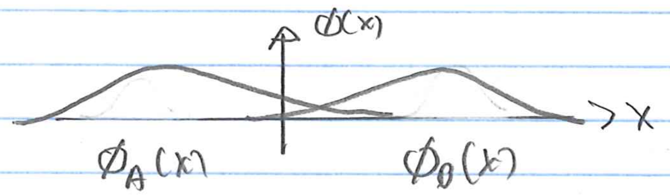
Integer spin $\Leftrightarrow$ symmetric, Bose Statistics

$\frac{1}{2}$-integer spin $\Leftrightarrow$ anti-symmetric Fermi statistics
Pauli-exclusion

Exchange density and exchange "force"

Cosider two spatial wave functions

$$\phi_A(x) \;,\; \phi_B(x)$$



spatially separated state with overlap

then 1, 2 particle labels

$$\psi_\pm = \frac{1}{\sqrt{2}}\left(\phi_A(x_1)\phi_B(x_2) \pm \phi_B(x_1)\phi_A(x_2)\right)$$

probability of finding particles in volume $dx_1 dx_2$

$$|\psi_\pm|^2 dx_1 dx_2$$

$$|\psi_\pm|^2 = \frac{1}{2}\Big[|\phi_A(x_1)|^2|\phi_B(x_2)|^2 + |\phi_A(x_2)|^2|\phi_B(x_1)|^2$$
$$\pm 2\,\mathrm{Re}\{\phi_A(x_1)\phi_B(x_2)\phi_A^*(x_2)\phi_B^*(x_1)\}\Big]$$

$\{\}$ is exchange density. Acts like "force" that pushes fermions apart, pulls bosons together.

For ϕ_A, ϕ_B well separated, overlap is negligible and exchange density is negligible

Wave function product of (space)(spin)
Symmetizing with

$$P_{12} = P_{12}^{space} P_{12}^{spin}$$

Can be simply generalized to more
function states like isospin.

Then two electron state ψ space, χ spin
properly antisymmetrized can be either

$$\psi_S \chi_A \quad \text{or} \quad \psi_A \chi_S$$

<u>Young Tableau</u> (1901) method for deriving
irreducible decomposition of SU(n).

Start with SU(2) spin. define single
particle states as

[1] spin up [2] spin down

So [ ] represents <u>doublet</u>

[ ][ ] symmetric tableau is <u>triplet</u>
count on

[1][1] no [2][1] which would
double count

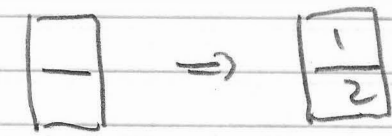
[1][2]

[2][2]

and

[ ]
[ ] completely antisymmetric <u>singlet</u>

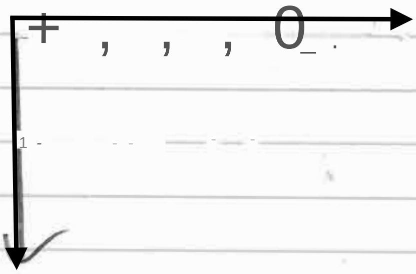
anti symmetric



only 1 possibility
singlet

general rule:

cannot decrease



must increase

electron tableau

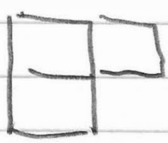

⇒

1 1 1
1 1 2
1 2 2
2 2 2

} dim 4
$j = 3/2$
symmetric

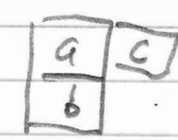
not possible

mixed symmetry



1 1 / 2   1 2 / 2   two states

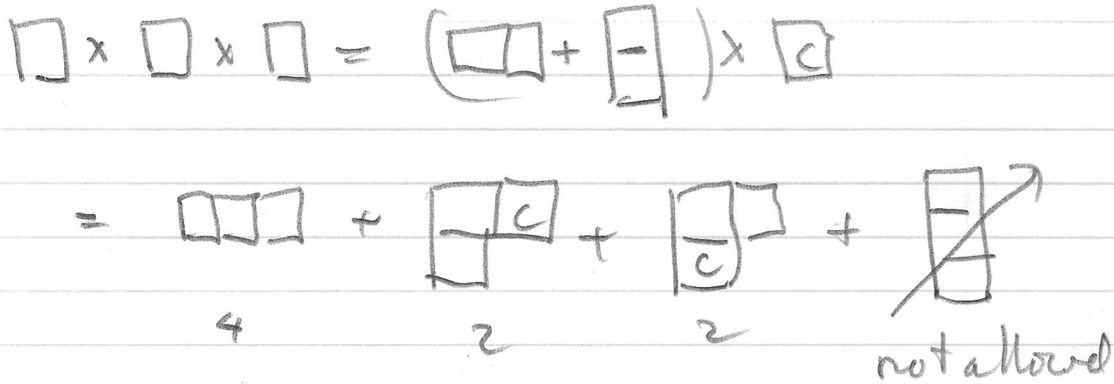
two possibilities for electrons



antisymmetric a, b

symmetric a, b

product of 3 SU(2) states!



more sophisticated methods exist
for getting multiplicity of irreducible
representation (irrep) in SU(n)

What exactly are two different
mixed symmetry irreps?

For SU(2), we do not really need Young Tableau because decomposition to irreducible representations (irreps) is easy:

$$|j_1-j_2| \le j \le j_1+j_2$$

$\underset{\sim}{2} \times \underset{\sim}{2} \times \underset{\sim}{2}$ or in highest j_z notation,

$$\tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = (1+0) \times \tfrac{1}{2}$$

$$= 1 \times \tfrac{1}{2} + \tfrac{1}{2} = \underset{\sim}{3/2} + \underset{\sim}{\tfrac{1}{2}} + \underset{\sim}{1/2} = \underset{\sim}{4} + \underset{\sim}{2} + \underset{\sim}{2}$$

What are two states?

use notation $|1/2, 1/2\rangle = u$, $|\tfrac{1}{2}, -1/2\rangle = d$

$\underset{\sim}{3/2}$ states completely symmetric

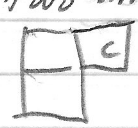
$$|3/2, 3/2\rangle = uuu$$

$$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}}(uud + udu + duu)$$

$$|3/2, -1/2) = \frac{1}{\sqrt{3}}(ddu + dud + udd)$$

$$|3/2, -3/2) = ddd$$

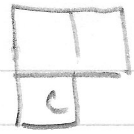
1/2 mixed symmetry, first two antisymmetrized
antisymmetrized



tableau

$$|1/2, 1/2\rangle = \frac{1}{\sqrt{2}}(ud - du)\,u$$

$$|1/2, -1/2\rangle = \frac{1}{\sqrt{2}}(ud - du)\,d$$

Other 1/2 mixed symmetry 

We need Clebsch Gordon

 $= |1,0\rangle = \frac{1}{\sqrt{2}}(ud+du)$

 $= |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = u$ or d

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,1\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\,(uu)d - \sqrt{\tfrac{1}{3}}\,\tfrac{1}{\sqrt{2}}(ud+du)u$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - \sqrt{\tfrac{2}{3}}\,|1,-1\rangle|\tfrac{1}{2}\,{+\tfrac{1}{2}}\rangle$$

$$= \sqrt{\tfrac{1}{3}}\,\tfrac{1}{\sqrt{2}}(ud+du)d - \sqrt{\tfrac{2}{3}}\,(dd)u$$

states are all properly orthonormal

Very briefly, SU(3) "Eight-fold way"

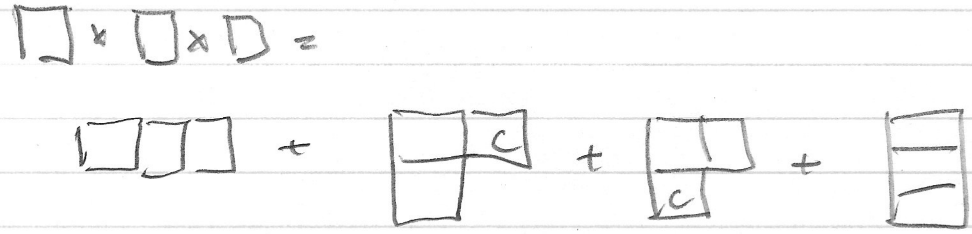
Gell-mann symmetry of 3 quark flavors

		≈ mass (meV)		
up	u	2.5	} isospin	$SU_F(3)$ flavor
down	d	5.0		
strange	s	95.0		

Symmetry only approximate because of heavy strange quark. But good because baryon (strongly interacting fermions like proton) are much larger

$M_p \approx 1000$ meV

with Young tableau, irreps of 3 quark flavors



dim	10	8	8	1
spin	3/2	1/2	1/2	0

mixed symmetry

$J = 1/2$ octet

$S=0$ n   p $I = 1/2$ Σ^0: uds symmetric

Λ: uds anti-symmetric

$S=-1$ Σ^- Σ^0, Λ Σ^+ $I = 1$

$S=-2$ Ξ^- Ξ^0 $I = 1/2$ "cascade"

isospin axis: -1, $-1/2$, 0, $1/2$, 1 → Isospin

$J = 3/2$ decuplet

$S=0$ Δ^- Δ^0 Δ^+ Δ^{++} $I = 3/2$

$S=-1$ Σ^{*-} Σ^{*0} Σ^{*+} $I = 1$

$S=-2$ Ξ^{*-} Ξ^{*0} $I = 1/2$

$S=-3$ Ω^- $I = 0$

isospin axis: $-3/2$, -1, $-1/2$, 0, $1/2$, 1, $+3/2$ → isospin

triply strange spin-3/2 Baryon
predicted by Gell-Mann.

discovered at Brookhaven shortly after (1964)

Corners of decuplet have problem with
Fermi statistics :

$\Omega^- = s\uparrow s\uparrow s\uparrow$ 3 fermions in same state

Introduce new quantum number color

red	r	$SU_c(3)$
blue	b	
green	g	

then

$\Omega^- = (s\uparrow r)(s\uparrow b)(s\uparrow g)$

Completely antisymmetric in color - color singlet

Only color singlet states exist as particles

$SU_c(3)$ — theory of QCD, generalization
of QED with 3 charges

QED is U(1) symmetry one generator, gauge field A^μ
photon

QCD SU(3) has $3^2-1 = 8$ generators
8 gauge fields, 8 gluons
and weak interaction, weak isospin $SU_w(2)$
has $2^2-1 = 3$ generators 3 gauge fields
W^+, Z, W^-