Photons

Recall from 521, lec. 2 Lagrangian for charged particle starts with Lorentz force:

$$\vec{F} = q\left(\vec{\mathcal{E}} + \frac{1}{c}\vec{v}\times\vec{B}\right)$$

potentials
$$\vec{\mathcal{E}} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Gauge transformation: χ arbitrary function of $\vec{x}, t$,

$$\phi' = \phi + \frac{1}{c}\frac{\partial\chi}{\partial t} \quad ; \quad \vec{A}' = \vec{A} - \vec{\nabla}\chi$$

leaves $\vec{\mathcal{E}}, \vec{B}$ unchanged

Lagrangian is

$$\mathcal{L} = \frac{1}{2}mv^2 - \left[q\phi - \frac{q}{c}\vec{v}\cdot\vec{A}\right]$$

velocity dependant potential

Legendre transformation to Hamiltonian $(q=-e)$

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 - e\phi + H_{Em}$$

In Q.M., we need potentials for interaction between particle and field.

$$H_{Em} = \frac{1}{8\pi}\int d^3r \left(|\vec{E}|^2 + |\vec{B}|^2\right)$$ field energy

<u>Photon</u> - quantum of the vector potential field $\vec{A}$.

Simplest to choose Coulomb (transverse) gauge.

$$\vec{\nabla}\cdot\vec{A} = 0$$

Then Maxwell's equations are $\qquad \rho$ charge density
$\qquad \vec{J}$ current density

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}\right) = -4\pi\rho$$

$$\nabla^2\vec{A} - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) - \vec{\nabla}\left(\cancelto{0}{\vec{\nabla}\cdot\vec{A}}\right) = -\frac{4\pi\vec{J}}{c}$$

<u>Note</u> - not manifestly Lorentz Invariant

$$\phi(\vec{r},t) = \int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

Connects ϕ to ρ at same instant in time.

<u>Free field equation of motion</u>

$\rho = 0$, $\vec{J} = 0$ everywhere, $\phi = 0$ and

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = 0$$

plane wave solution

$$\vec{A} = \vec{A}_0 \exp(i\vec{k}\cdot\vec{r} - \omega t)$$

$\omega = c|\vec{k}|$

gauge condition, $\vec{\nabla}\cdot\vec{A} = (\vec{k}\cdot\vec{A}_0)\exp(i\vec{k}\cdot\vec{r} - \omega t) = 0$

implies $\quad \vec{k} \perp \vec{A}_0 \qquad$ transverse gauge

Photon has two polarization states

$$\vec{\varepsilon}_i \cdot \vec{k} = 0 \qquad i = 1,2$$

either linear or circular polarizations.

Circular polarization states have angular momentum and states

$$|1, \pm 1\rangle$$

even though $l = 1$, longitudinal polarization state $|1,0\rangle$ does not exist.
Forbidden by gauge invariance and is result of massless photon.

mass term in Lagrangian would be

$$m^2 |\vec{A}|^2 \quad \text{not gauge invariant}$$

Then free field Hamiltonian ~~in~~ Coulomb gauge is

$$H_{EM} = \frac{1}{8\pi} \int d^3r \left(|\vec{E}|^2 + |\vec{B}|^2 \right)$$

$$= \frac{1}{8\pi} \int d^3r \left[\frac{1}{c^2} \left(\frac{\partial \vec{A}}{\partial t} \right)^2 + \left(\vec{\nabla} \times \vec{A} \right)^2 \right]$$

For quantum theory, we must quantize each field degree of freedom, (d.o.f)

$$\vec{A}(\vec{r},t) = \int d^3k \left\{ \vec{A}_{\vec{k}} \exp\left[i(\vec{k}\cdot\vec{r}-\omega t)\right] + \vec{A}_{\vec{k}}^{*} \exp\left[-i(\vec{k}\cdot\vec{r}-\omega t)\right] \right\}$$

infinity of d.o.f., one for each $\vec{k}$.

Box Quantization

Quantize in box of Volume $V = L^3$ and then take $L \to \infty$.

$$e^{ik_x x} = e^{ik_x(x+L)}, \quad k_x = \frac{2\pi n_x}{L} \quad n_x = 0, \pm 1, \pm 2, \ldots$$

$$\vec{A}_{\vec{k}} = C_{\vec{k},\lambda}\, \vec{\varepsilon}(\vec{k},\lambda) \qquad \lambda \text{ polarization index } 1, 2$$

$\vec{\varepsilon}$ unit polarization vector

$$\vec{k} = \hat{x}\left(\frac{2\pi n_x}{L}\right) + \hat{y}\left(\frac{2\pi n_y}{L}\right) + \hat{z}\left(\frac{2\pi n_z}{L}\right)$$

and $\vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{k} = 0$

General solution. Note $\vec{A}$ is real
take $n_x > 0,\ n_y > 0,\ n_z > 0$

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\substack{\vec{k} \\ n_x, n_y, n_z}} \sum_{\lambda=1}^{2} \left[C_{\vec{k},\lambda}\, \vec{\varepsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + C^{*}_{\vec{k},\lambda}\, \vec{\varepsilon}^{\,*}(\vec{k},\lambda)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \right]$$

$\vec{\varepsilon}^{\,*}$ for circular polarizations

Modes are orthonormal:

$$\int_{-L/2}^{L/2} dx \left(\frac{e^{in'\frac{2\pi x}{L}}}{\sqrt{L}}\right)\left(\frac{e^{in\frac{2\pi x}{L}}}{\sqrt{L}}\right)$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} \exp\left(i(n-n')\frac{2\pi x}{L}\right) dx$$

$$= \begin{cases} \left(\frac{1}{L}\right)\frac{L}{(n-n')\pi}\sin[(n-n')\pi] = 0 & n \neq n' \\ 1 & n = n' \end{cases}$$

$$= \delta_{n,n'}$$

In three dimensions,

$$\int d^3r \left(\frac{e^{i(\vec{k}'\cdot\vec{r}-\omega' t)}}{\sqrt{V}}\right)^* \left(\frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{\sqrt{V}}\right)$$

$$= \delta(\vec{k}-\vec{k}')$$

Hamiltonian:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(C_{\vec{k},\lambda}\,\vec{\varepsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

$$= \frac{i\omega}{c}\, C_{k,\lambda}\,\vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

and $\vec{\nabla}\times\left(C_{k,\lambda}\vec{\varepsilon}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) = i\vec{k}\times\vec{\varepsilon}\; C_{k,\lambda}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

together with orthogonality and $|\vec{k}| = \omega/c$

$$\boxed{H_{EM} = \frac{1}{2\pi}\sum_{\vec{k},\lambda} k^2 \left| C_{\vec{k},\lambda}\right|^2}$$

Free field quantization analog of $[X,p]=i\hbar$?

put time dependence into coefficients C as

$$C_{\vec{k},\lambda}(t) \equiv C_{k,\lambda}(0)\, e^{-i\omega t}$$

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}}\sum_{\vec{k},\lambda}\left[C_{\vec{k},\lambda}(t)\,\vec{\varepsilon}\, e^{i\vec{k}\cdot\vec{r}} + C^*_{\vec{k},\lambda}(t)\,\vec{\varepsilon}^{\,*}\, e^{-i\vec{k}\cdot\vec{r}}\right]$$

C's satisfy:

$$\frac{d^2}{dt^2} C_{\vec{k},\lambda} = -\omega^2 C_{\vec{k},\lambda}$$

Define:

$$q_{\vec{k},\lambda}(t) \equiv \frac{1}{c\sqrt{4\pi}}\left(c_{\vec{k},\lambda} + c^*_{\vec{k},\lambda}\right)$$

$$p_{\vec{k},\lambda}(t) \equiv \frac{-i\omega}{c\sqrt{4\pi}}\left(c_{\vec{k},\lambda} - c^*_{\vec{k},\lambda}\right)$$

then $\quad \dot{q} = p \qquad$ for each $\vec{k},\lambda$

$$\dot{p} = -\omega^2 q$$

and

$$H_{EM} = \sum_{\vec{k},\lambda} \frac{1}{2}\left[p^2_{\vec{k},\lambda} + \omega^2 q^2_{\vec{k},\lambda} \right]$$

q, p are canonically conjugate variables:

$$\left.\begin{aligned} \frac{\partial H}{\partial q} &= \omega^2 q = -\dot{p} \\ \frac{\partial H}{\partial p} &= \dot{q} \end{aligned}\right\} \text{ Hamilton's equations}$$

Quantize with equal time commutators

$$\left[\hat{q}_{\vec{k},\lambda}(t), \hat{p}_{\vec{k}',\lambda'}(t) \right] = i\hbar\, \delta_{\vec{k}\vec{k}'}\, \delta_{\lambda\lambda'}$$

Define creation, annihilation operators:

recall $\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right) = \sqrt{\frac{\omega}{2\hbar}}\left(\sqrt{m}\hat{X} + \frac{i}{\omega\sqrt{m}}\hat{P}\right)$
for simple harmonic oscillator in 1D.

So $\hat{a}_{\vec{k},\lambda} \equiv \sqrt{\frac{\omega}{2\hbar}}\left(\hat{q}_{\vec{k},\lambda} + \frac{i}{\omega}\hat{p}_{\vec{k},\lambda}\right)$

Then comparing to expansion in terms of C,

$$C_{\vec{k},\lambda} \longrightarrow c\sqrt{\frac{2\pi\hbar}{\omega}}\,\hat{a}_{\vec{k},\lambda}$$

We then have $\vec{A}$ as a quantum field:

$$\vec{A}(\vec{r},t) = \frac{c}{\sqrt{V}}\sum_{\vec{k},\lambda}\sqrt{\frac{2\pi\hbar}{\omega}}\left(\hat{a}_{\vec{k},\lambda}(t)\,\vec{\varepsilon}\,e^{i\vec{k}\cdot\vec{r}} + \hat{a}^{+}_{\vec{k},\lambda}(t)\,\vec{\varepsilon}^{*}e^{-i\vec{k}\cdot\vec{r}}\right)$$

where $\left[a_{\vec{k},\lambda}(t),\ a^{+}_{\vec{k},\lambda}(t)\right] = \delta(\vec{k}-\vec{k}')\,\delta(\lambda-\lambda')$

Hamiltonian:

$$H_{EM} = \frac{1}{8\pi}\int d^3r\left[\left(-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)^2 + \left(\vec{\nabla}\times\vec{A}\right)^2\right]$$

$$= \frac{1}{2}\sum_{\vec{k},\lambda}\hbar\omega\left(aa^{+} + a^{+}a\right)_{\vec{k},\lambda}$$

$$= \sum_{\vec{k},\lambda}\hbar\omega\left(a^{+}_{\vec{k},\lambda}a_{\vec{k},\lambda} + \frac{1}{2}\right)$$

where # photons in mode is $a^{+}_{\vec{k},\lambda}\,a_{\vec{k},\lambda}$

<u>Can get Hamiltonian from</u> (supress $\vec{k}, \lambda$)

$$a = \sqrt{\frac{\omega}{2\hbar}}\left(q + \frac{i}{\omega}p\right) \qquad a^+ = \sqrt{\frac{\omega}{2\hbar}}\left(q - \frac{i}{\omega}p\right)$$

$$q = \sqrt{\frac{\hbar}{2\omega}}\,(a + a^+) \qquad p = i\sqrt{\frac{\hbar\omega}{2}}\,(a^+ - a)$$

$$H = \frac{1}{2}\sum(p^2 + \omega^2 q^2) = \frac{1}{2}\left(\frac{\hbar\omega}{2}\right)\left[(a+a^+)^2 - (a^+-a)^2\right]$$

$$= \sum \frac{\hbar\omega}{4}\left[a^2 + a^{+2} + aa^+ + a^+a - (a^2 + a^{+2} - a^+a - aa^+)\right]$$

$$= \sum \frac{\hbar\omega}{2}(a^+a + aa^+) = \sum \hbar\omega\left(a^+a + \frac{1}{2}\right)$$

Vacuum Energy

$$H_{vac} = \underset{\substack{\uparrow \\ \text{polarization}}}{2} \sum_{\vec{k}} \left(\frac{\hbar\omega}{2}\right) \qquad \omega = c\sqrt{k_x^2 + k_y^2 + k_z^2}$$

density of photon states is:

$$\frac{V d^3k}{(2\pi)^3} = \frac{V(4\pi)k^2 dk}{(2\pi)^3} = \frac{2V}{(2\pi)^2}\frac{1}{c^3}\omega^2 d\omega$$

$$H_{vac} = \sum_{\vec{k}} \hbar\omega \rightarrow 2\int \frac{V d^3k}{(2\pi)^3}(\hbar\omega)$$

$$\frac{2V}{4\pi^2}\frac{1}{c^3}\hbar\int_0^{\omega_{cut}} \omega^3 d\omega = \frac{\hbar V}{2\pi^2}\frac{1}{c^3}\omega_{cut}^4$$

Vac energy density $\rho_{vac} \propto \omega_{cut}^4$

In G.R., vacuum energy density is equivalent to cosmological constant

$$\Lambda = \frac{8\pi G}{c^2}\rho_{vac} \qquad G = \text{Newton's constant}$$

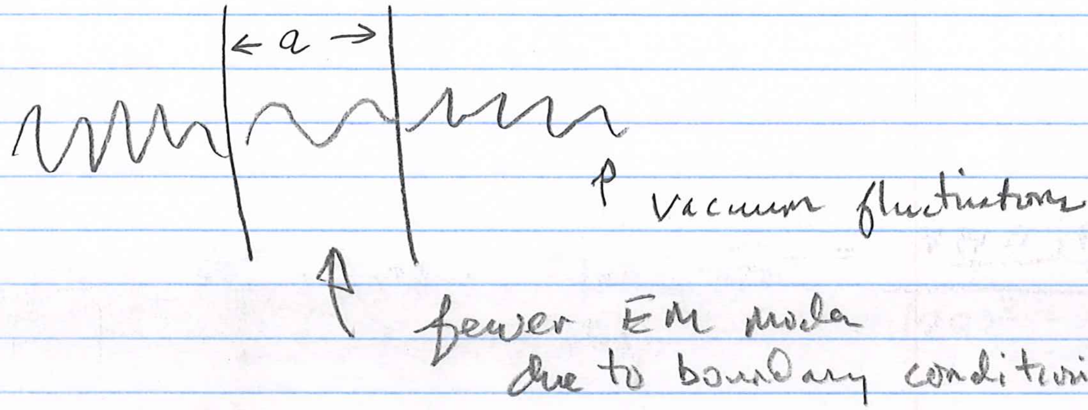
with cut-off at planck scale, $\hbar\omega_{cut} = 10^{19}$ GeV

$$\Lambda_{theory} = 10^{120}\Lambda_{obs}$$

where Λ_{obs} is observed acceleration of Hubble expansion.

EM Vacuum gives rise to **Casimir Effect**

Attractive force between electrically neutral conducting plates (area A)



$$\frac{F}{A} = \frac{-\pi^2}{240}\frac{\hbar c}{a^4} = -\frac{0.016 \text{ dyn}/\text{cm}^2}{a^4_{\mu m}}$$

minus means attractive

1 dyn = wt of 0.001 g

Photon states

Vacuum state $|0\rangle$ no photons

$$a_{\vec{k},\lambda}|0\rangle = 0$$

$$|\vec{k},\lambda\rangle = a^{+}_{\vec{k},\lambda}|0\rangle$$

$$|n_{\vec{k}},\lambda\rangle = \frac{(a^{+}_{k,\lambda})^{N}}{\sqrt{N!}}|0\rangle \qquad n\text{-photon state}$$

momentum

$$\hat{\vec{p}} = \frac{1}{4\pi c}\int d^3r\ \vec{E}\times\vec{B} = \sum_{\vec{k},\lambda}\hbar\vec{k}\,a^{+}_{\vec{k},\lambda}a_{\vec{k},\lambda}$$

$$\hat{\vec{p}}\,|n_{\vec{k}},\lambda\rangle = \hbar\vec{k}\,n\,|n_{\vec{k}}\rangle$$

photons carry momentum

L, R circular polarization states

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\vec{k},1\rangle \pm i\,|\vec{k},2\rangle\right)$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left(a^{+}_{\vec{k},1} \pm i\,a^{+}_{k,2}\right)}_{a^{+}_{\vec{k}\pm}}|0\rangle$$

Angular momentum operator

$$\hat{\vec{J}} = \int d^3r \, \vec{r} \times \left(\frac{\vec{E} \times \vec{B}}{4\pi c} \right)$$

Can show

$$\frac{\vec{J} \cdot \vec{k}}{\hbar} \, | \pm \rangle = \pm \hbar \, | \pm \rangle$$

photons carry angular momentum

Hamiltonian for Atom + EM field

$$H_0 = \frac{p^2}{2\mu} - \frac{e^2}{r} + \frac{1}{8\pi}\int d^3r\left[\left(\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)^2 + (\vec{\nabla}\times\vec{A})^2\right]$$

atom free field

Interaction is

$$H_1 = \frac{1}{2m}\left(\vec{P} + \frac{e}{c}\vec{A}\right)^2$$

$$H_1\psi = \frac{e}{2mc}\vec{A}\cdot\left(\frac{\hbar}{i}\vec{\nabla}\right)\psi + \frac{e}{2mc}\frac{\hbar}{i}\vec{\nabla}\cdot(\vec{A}\psi) + \frac{e^2}{2mc^2}|\vec{A}|^2\psi$$

where $\vec{\nabla}\cdot(\vec{A}\psi) = \underbrace{(\nabla\cdot\vec{A})}_{0 \text{ in Coulomb}}\psi + \vec{A}\cdot\vec{\nabla}\psi$

2^{nd} piece adds to first term giving

$$H_1(t) = \frac{e}{mc}\vec{A}(t)\cdot\left(\frac{\hbar}{i}\vec{\nabla}\right) + \frac{e^2}{2mc^2}|\vec{A}(t)|^2$$

perturbation is time dependent.