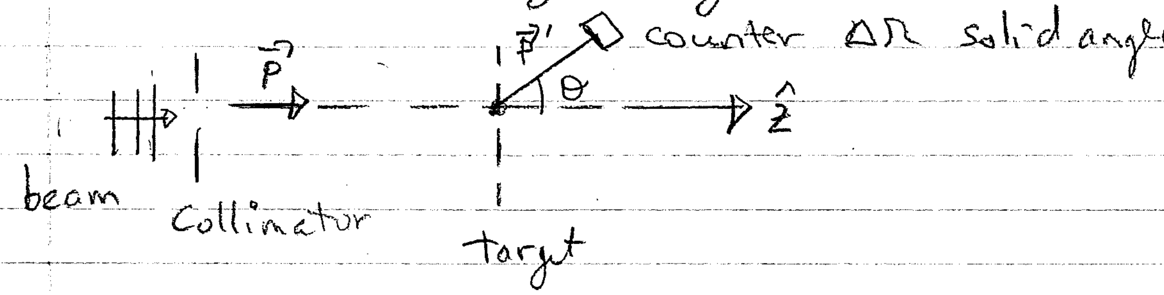
Phys 522

Scattering Theory I



Collimated beam of particles, approximately plane wave

$\vec{p} = \hbar \vec{k}\,, \quad \phi_{\vec{k}}(\vec{r}) = \langle \vec{r} \,|\, \vec{k} \rangle \equiv e^{i\vec{k}\cdot\vec{r}}$

normalized as $\quad \langle \vec{k}' \,|\, \vec{k} \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$

Quantity of physical interest is the differential cross section

$$d\sigma = \left(\frac{d\sigma}{d\Omega}\right) d\Omega \equiv \frac{\text{Rate scattered into } d\Omega \text{ @ } \theta, \phi}{\text{incident flux}}$$

flux $\equiv$ # incident particles / area / time
measured, for example, as $\dfrac{\text{current}}{q\,(\text{collimator area})}$

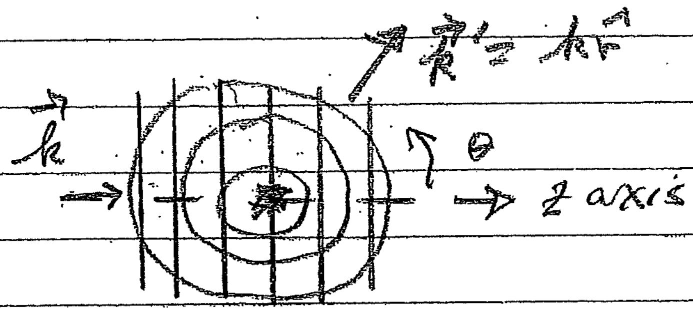
Dividing by flux leaves quantity intrinsic to interaction (scattering potential).

σ is dimensionally an area, area $\perp$ to beam.
Lorentz invariant under boosts in beam direction.

Elastic Scattering

In the cm frame (or for a fixed force center) only the direction of the momentum changes in the collision: $|\vec{k}'| = |\vec{k}|$

Time independent description:



We cannot observe the scattering in the shaded region of size $\lambda = \frac{1}{k}$. Observations are in the asymptotic region $r \gg \lambda$

$$\psi \underset{r\to\infty}{\sim} \psi_{in} + \psi_{sc}$$

where $\psi_{in} = A e^{ikz}$ incident plane wave

$$\psi_{sc} \sim A \frac{e^{ikr}}{r} f(\theta,\phi) \quad \text{outgoing spherical wave}$$

We will derive this asymptotic form rigorously later. $f(\theta,\phi)$ is called the scattering amplitude.

Note: probability is conserved by destructive interference in the forward ($\theta = 0$) direction. (optical theorem) recall probability flux:

$$\vec{j} = \frac{\hbar}{2\mu i}\left(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = \frac{\hbar}{\mu} \mathrm{Im}\left(\psi^*\vec{\nabla}\psi\right)$$

$$\vec{j}_{in} = \frac{\hbar k}{\mu} |A|^2 \hat{z}$$

$$\vec{j}_{sc} \underset{r\to\infty}{\sim} \frac{\hbar k}{\mu r^2} |A|^2 |f|^2 \hat{r}$$

$$\left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{(\vec{j}_{sc}\cdot\hat{r})\, r^2 d\Omega}{|\vec{j}_{in}|} = |f|^2 d\Omega$$

$$\frac{d\sigma}{d\Omega} = |f|^2 \quad \text{scattering amplitude squared}$$

Note: since A cancels in the ratio of fluxes we take $A=1$ from now on.

Optical theorem: $\mathrm{Im} f(\theta=0) = \frac{k\,\sigma_{tot}}{4\pi}$

$$\sigma_{tot} = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

Formal Development

$H = H_0 + V$

$H_0 = \frac{p^2}{2m}$ free particle, energy eigenstate

$$H_0 |\phi\rangle = E|\phi\rangle \qquad E = \frac{\hbar^2 k^2}{2m}$$

Complete solution,

$$H|\psi\rangle = (H_0 + V)|\psi\rangle = E|\psi\rangle \qquad (1)$$

We can solve formally if we can define inverse operator,

$$(E - H_0)|\psi\rangle = V|\psi\rangle \qquad (2)$$

$$|\psi\rangle = \left[E - H_0\right]^{-1} V|\psi\rangle + |\phi\rangle$$

apply $(E - H_0)$ to each side gives back original equation.

This is formal solution since $|\psi\rangle$ appears on both sides. The addition of $|\phi\rangle$ takes into account that at large distances, solution is free particle.

In position space, the plane wave

$\phi_k(\vec{r}) = \langle \vec{r} | \vec{k} \rangle = e^{i\vec{k}\cdot\vec{r}}$ where it is convenient to use the normalization of the plane wave

$$\langle \vec{k}' | \vec{k} \rangle = (2\pi)^3 \delta^3(\vec{k}' - \vec{k})$$

In position space: $\int d^3r''|\vec{r}''\rangle\langle\vec{r}''|$ $\int d^3r'|\vec{r}'\rangle\langle\vec{r}'|$

$$\langle\vec{r}|\psi\rangle = \langle\vec{r}|\phi\rangle + \langle\vec{r}|(E-H_0)^{-1}V|\psi\rangle$$

$$\langle\vec{r}''|V|\vec{r}'\rangle = V(\vec{r}')\,\delta^3(\vec{r}''-\vec{r}')$$

becomes

$$\psi(\vec{r}) = \phi_{\vec{k}}(\vec{r}) + \int d^3r'\,\langle\vec{r}|(E-H_0)^{-1}|\vec{r}'\rangle V(\vec{r}')\psi(\vec{r}')$$

then define $\langle\vec{r}|(E-H_0)^{-1}|\vec{r}'\rangle \equiv \frac{2m}{\hbar^2}G(\vec{r},\vec{r}')$
where the Green's function satisfies

$$(\nabla^2+k^2)\,G(\vec{r},\vec{r}') = \delta^3(\vec{r}-\vec{r}')$$

this works because by substitution we get back

$$(\nabla^2+k^2)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\psi(\vec{r})$$

or $(-H_0+E)|\psi\rangle = V\psi$ or $(H_0+V)|\psi\rangle = E|\psi\rangle$
original eigenvalue equation.
The Green's function is evaluated by $\vec{r}'=0$
Fourier transform, $G(\vec{r},\vec{r}') = G(\vec{r}-\vec{r}') \longrightarrow G(\vec{r})$

$$\delta(\vec{r}) = \left(\frac{1}{2\pi}\right)^3\int d^3q\, e^{i\vec{q}\cdot\vec{r}}$$ ← Fourier transform

$$G(\vec{r}) = \left(\frac{1}{2\pi}\right)^{3/2}\int d^3q\, e^{i\vec{q}\cdot\vec{r}}\,\tilde{G}(q)$$

$$\left(\frac{1}{2\pi}\right)^{3/2}\int d^3q\,(-q^2+k^2)\,e^{i\vec{q}\cdot\vec{r}}\,\tilde{G}(q) = \left(\frac{1}{2\pi}\right)^3\int d^3q\, e^{i\vec{q}\cdot\vec{r}}$$

giving $\tilde{G}(q) = \left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{1}{k^2-q^2}\right)$
note singular for $k=q$

<u>Green's function or propagator</u>

$$G(\vec{r},\vec{r}') = \frac{\hbar^2}{2m} \langle \vec{r} | \left(E - H_0\right)^{-1} | \vec{r}' \rangle$$

$$\int d^3q\, |\vec{q}\rangle\langle\vec{q}| \qquad \int d^3q'\, |\vec{q}'\rangle\langle\vec{q}'|$$

where $|\vec{q}\rangle$ are plane wave states,

$$H_0 |q\rangle = \frac{\hbar^2 q^2}{2m} |q\rangle$$

$$\langle \vec{q}' | \vec{q} \rangle = (2\pi)^3 \delta^3(\vec{q}' - \vec{q})$$

so $\langle \vec{q} | (E - H_0)^{-1} | \vec{q}' \rangle = \frac{1}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right) \frac{\delta^3(\vec{q}' - \vec{q})}{k^2 - q^2}$

where $E = \frac{\hbar^2 k^2}{2m}$

We get, with $\langle \vec{r} | \vec{q} \rangle = e^{i\vec{q}\cdot\vec{r}}$

$$G(\vec{r},\vec{r}') = \left(\frac{1}{2\pi}\right)^3 \int d^3q \, \frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}}{k^2 - q^2}$$

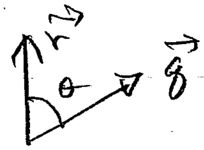
$$= \left(\frac{1}{2\pi}\right)^{3/2} \int d^3q \, e^{i\vec{q}\cdot(\vec{r}-\vec{r}')} \tilde{G}(q)$$

with $\tilde{G}(q) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{k^2 - q^2}$

Fourier transform

Then

$$G(\vec r) = \left(\frac{1}{2\pi}\right)^3 \int d^3q\, e^{i\vec q\cdot\vec r}\left(\frac{1}{k^2-q^2}\right)$$

evaluate the integral 

$\vec q\cdot\vec r = qr\cos\theta \equiv qrc$

$$G(\vec r) = \left(\frac{1}{2\pi}\right)^3(2\pi)\int_0^\infty q^2\,dq\left(\frac{1}{k^2-q^2}\right)\int_{-1}^{+1}dc\, e^{iqrc}$$

$$= \left(\frac{1}{2\pi}\right)^2\frac{1}{r}\int_0^\infty \frac{q\,dq}{k^2-q^2}\,2\sin(qr)$$

$$= \frac{1}{4\pi^2}\left(\frac{1}{ir}\right)\int_0^\infty\frac{q\,dq}{k^2-q^2}\left(e^{iqr}-e^{-iqr}\right)$$

→ let $q=-q'$

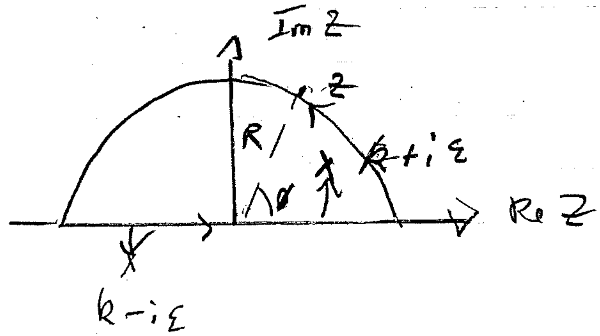
$$\left(\frac{1}{4\pi^2}\right)\frac{1}{ir}\left[\int_0^\infty \frac{q\,dq}{k^2-q^2}\,e^{iqr} - \int_0^{-\infty}\frac{q\,dq}{k^2-q^2}\,e^{iqr}\right]$$

$$= \frac{i}{4\pi^2}\left(\frac{1}{r}\right)\int_{-\infty}^{\infty} \frac{q\,dq\,e^{iqr}}{q^2-k^2}$$

may be evaluated by contour integration

$$q^2-k^2 = (q+k+i\varepsilon)(q-k-i\varepsilon)$$

$q\to z = Re^{i\phi}$



$q \to z = Re^{i\phi}$ contour integral. Enclose pole at $z = k + i\epsilon$ by closing contour in upper half plane,

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \oint \frac{z\,dz\,e^{iqr}}{(z+k+i\epsilon)(z-k-i\epsilon)} = 2\pi i\left(\text{Residue at } z = k + i\epsilon\right)$$

Residue $= \frac{k}{2k} e^{ikr} = \frac{1}{2} e^{ikr}$, so integral $= \pi i e^{ikr}$

Integral along real axis gives desired integral,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{R\,dR\,e^{iRr}}{(R+k+i\epsilon)(R-k-i\epsilon)} = \int_{-\infty}^{\infty} \frac{q\,dq\,e^{iqr}}{(q+k+i\epsilon)(q-k-i\epsilon)}$$

Integral over semicircle at fixed R, $dz = Ri\,d\phi$

$z = Re^{i\phi}$

$$\lim_{R\to\infty} \int^{\pi} \frac{R e^{2i\phi} i\,d\phi\, e^{izr}}{R^2 e^{2i\phi}} = \lim_{R\to\infty} \int_0^{\pi} \frac{i\,d\phi}{R}\, e^{iRr\cos\phi}\, e^{-Rr\sin\phi} = 0$$

with $e^{izr} = \exp(iRr\cos\phi)\exp(-Rr\sin\phi)$

$$\text{So } G_+(r) = \frac{i}{4\pi^2}\left(\frac{1}{r}\right)\pi i e^{ikr} = -\frac{1}{4\pi}\frac{e^{ikr}}{r}$$

We see that choice of pole gives <u>outgoing</u> spher<u>ical</u> <u>wave</u>.

We arrive at

$$\psi(\vec r) = \phi_k(\vec r) - \frac{m}{2\pi\hbar^2}\int d^3r'\, \frac{e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|} V(\vec r\,')\psi(\vec r\,')$$

Asymptotic Limit

$$|\vec{r}-\vec{r}'| = \left(r^2+r'^2-2\vec{r}\cdot\vec{r}'\right)^{1/2} = r\left(1+\left(\frac{r'}{r}\right)^2-2\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

$$\approx r - \left(\frac{\vec{r}}{r}\right)\cdot\vec{r}'$$

then $k|\vec{r}-\vec{r}'| = kr - k\hat{r}\cdot\vec{r}' = kr - \vec{k}'\cdot\vec{r}'$

$$\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r}\, e^{-i\vec{k}'\cdot\vec{r}'}$$

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}\left[\frac{-m}{2\pi\hbar^2}\int d^3r'\, e^{-i\vec{k}'\cdot\vec{r}'}\,\psi(\vec{r}')\,V(\vec{r}')\right]$$

we see $f(\theta,\phi) = [\;]$ scattering amplitude

Convergence of integral requires $V(r) \xrightarrow[r\to\infty]{} 0$ faster than $1/r$.

Iterative solution of integral equation

Define transition operator T as

$$T|\phi\rangle = V|\psi\rangle$$

$$|\psi\rangle = |\phi\rangle + (E-H_0)^{-1}V|\psi\rangle = |\phi\rangle + (E-H_0)^{-1}T|\phi\rangle$$

operate with V

$$V|\psi\rangle = T|\phi\rangle = V|\phi\rangle + V(E-H_0)^{-1}T|\phi\rangle$$

operator equation for T

$$T = V + V(E-H_0)^{-1}T$$

with iterative solution

$$T = V + V(E-H_0)^{-1}V + V(E-H_0)^{-1}V(E-H_0)^{-1}V + \cdots$$

relate to scattering amplitude

$$f(\theta,\phi) = f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\, e^{-i\vec{k}'\cdot\vec{r}'}V(\vec{r}')\psi(\vec{r}')$$

with $\langle\vec{r}'|V|\vec{r}''\rangle = V(\vec{r}')\delta^3(\vec{r}'-\vec{r}'')$

$$f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int d^3r'\,\langle k'|\vec{r}'\rangle\langle\vec{r}'|V|\psi\rangle$$

$$= -\frac{m}{\;} \langle\vec{k}'|V|\psi\rangle = -\frac{m}{2\pi\hbar^2}\langle\vec{k}'|T|\vec{k}\rangle$$

f is the transition $|k\rangle \xrightarrow[T]{} |k'\rangle$

Inserting iterative solution gives
Born series:

$$f(\vec{k},\vec{k}') = \sum_{n=1}^{\infty} f^{(n)}(\vec{k},\vec{k}')$$

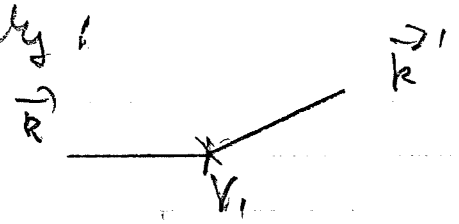
n is number of times V appears

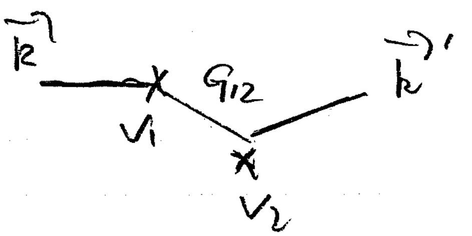
$$f^{(1)} = -\frac{m}{2\pi\hbar^2}\langle \vec{k}'|V|\vec{k}\rangle$$ Born approximation

$$f^{(2)} = \frac{-m}{2\pi\hbar^2}\langle \vec{k}'|V(E-H_0)^{-1}V|\vec{k}\rangle$$

etc,

Graphically:





G is propagator

Unpack $f^{(2)}$:

$$f^{(2)} = \frac{-m}{2\pi\hbar^2} \langle k' | \int d^3r' \, |r'\rangle\langle r'| V (E-H_0)^{-1} \int d^3r'' \, |r''\rangle$$
$$\times \langle r'' | V | \vec{k} \rangle$$

$$= \frac{-m}{2\pi\hbar^2} \int d^3r' \int d^3r'' \, \langle \vec{k}' | \vec{r}\,' \rangle V(r')$$
$$\times \underbrace{\langle \vec{r}\,' | (E-H_0)^{-1} | r'' \rangle}_{\frac{2m}{\hbar^2} G_+ (\vec{r}\,' - \vec{r}\,'')} V(r'') \langle \vec{r}\,'' | k \rangle$$

$$= \frac{-m^2}{\pi\hbar^4} \int d^3r' \, d^3r'' \, e^{-i\vec{k}' \cdot \vec{r}\,'} V(r') G_+(\vec{r}\,', \vec{r}\,'')$$
$$\times V(r'') \, e^{i\vec{k} \cdot \vec{r}\,''}$$

with $G_+(\vec{r}\,', \vec{r}\,'') = -\frac{1}{4\pi} \frac{e^{ik|\vec{r}\,' - \vec{r}\,''|}}{|\vec{r}\,' - \vec{r}\,''|}$

Evaluating the Born Amplitude

$$f^{(1)} = \frac{-m}{2\pi\hbar^2} \langle \vec{k}' | V | \vec{k} \rangle$$

$$= \frac{-m}{2\pi\hbar^2} \int d^3\vec{r}\,' \, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}\,'} V(\vec{r}\,')$$

$f^{(1)}$ is the Fourier transform of the potential.
let $\vec{q} = \vec{k} - \vec{k}'$ momentum transfer.
Angular integral ($c = \cos\theta$) for spherically symmetric V

$$f^{(1)}(\theta) = \frac{-m}{2\pi\hbar^2} \int_0^\infty r'^2 dr' \, V(r') \, (2\pi) \underbrace{\int_{-1}^{+1} dc \, e^{iqr'c}}_{\frac{2}{qr'} \sin(qr')}$$

$$\boxed{f^{(1)}(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty r' dr' \sin(qr') \, V(r')}$$

$$q^2 = |\vec{q}|^2 = k^2 + k^2 - 2k^2\cos\theta = 2k^2(1-\cos\theta)$$
$$= 4k^2 \sin^2 \theta/2$$

$$\boxed{q = 2k \sin \theta/2}$$ momentum transfer in C.M. frame

Note that integral diverges for Coulomb potential.

Validity of Born Approximation

Scattered wave is small,

$$\frac{|\psi_{sc}(\vec{r}=0)|}{|\psi_{in}(\vec{r}=0)|} \ll 1$$

To get $\psi_{sc}(\vec{r}=0)$, have to go back to before the asymptotic limit of G_+

$$\left|\psi^{(1)}_{sc}(0)\right| = \left|\frac{2m}{\hbar^2}\left(\frac{-1}{4\pi}\right)\int d^3r' \frac{e^{ikr'}}{r'} V(r')\, \phi_{\vec{k}}(\vec{r}')\right|$$

$$\psi_{in} = e^{i\vec{k}\cdot\vec{r}}\Big|_{\vec{r}=0} = 1 \qquad\qquad e^{i\vec{k}\cdot\vec{r}'} = e^{ikr'c}$$

$$\left|\frac{\psi_{sc}(0)}{\psi_{in}(0)}\right| = \frac{m}{2\pi\hbar^2}\left|\int_0^\infty r'^2 dr'\left(\frac{e^{ikr'}}{r'}\right)V(r')\, 2\pi \underbrace{\int_{-1}^{+1} dc\, e^{ikr'c}}_{\frac{2}{kr'}\sin(kr')}\right|$$

$$= \frac{2m}{\hbar^2 k}\left|\int_0^\infty dr'\, V(r')\, \underbrace{e^{ikr'}\sin(kr')}_{\frac{1}{2i}\left(e^{i2kr'}-1\right)}\right|$$

High energy limit $k \to \infty$, integral of rapidly oscillating $e^{i2kr'}$ gives zero.

Then at high energy (large k) the validity condition is

$$\frac{m}{\hbar^2 k}\left|\int_0^\infty dr\, V(r)\right| \ll 1$$

Assuming the integral converges we can define average V over range r_0,

$$\int_0^\infty dr\, V(r) = V_0 r_0$$

Then $\frac{m V_0 r_0}{\hbar^2 k} \ll 1$ see Cummins, eq. 18.94 valid for large k

For Coulomb potential we can set a minimum distance $1/k$ and screening distance a

$$\left|\int_{1/k}^{a} dr\, \frac{-Ze^2}{r}\right| = Ze^2 \ln(ka)$$

$$\frac{m}{\hbar^2 k}(Ze^2)\ln(ka) \ll 1$$ also valid for large k

See Sakurai, eq 7.2.15

In general, expect Born to be valid at large energy. However, at sufficiently high energy, higher order terms in expansion of f become large.

At low energy $k \to 0$ limit

$$e^{ikr'} \sin(kr') \approx kr'$$

Validity condition is

$$\frac{2m}{\hbar^2}\left|\int_0^\infty r\,dr\,V(r)\right| \ll 1$$

Again, if integral converges we can define an average potential V_0 over an effective range r_0,

$$\frac{2m}{\hbar^2} r_0^2 V_0 \ll 1$$

Comparing to high energy condition, we see that if Born is valid for low energy it is valid for all energies.

° For low energies, the method of partial waves is more generally valid

Born example.

$$V(r) = g\,\frac{e^{-r/r_0}}{r}$$

Yukawa potential with range $r_0 = \frac{1}{m_0}$ note definition of m_0

$$f^{(1)} = \frac{-2m}{\hbar^2 q}\int_0^\infty r'dr' \sin(qr')\,\frac{g}{r'}\,e^{-r'/r_0}$$

with $\sin qr = \frac{1}{2i}\left(e^{iqr} - e^{-iqr}\right)$

$$f^{(1)}(q^2) = \frac{-2mg}{\hbar^2(q^2+m_0^2)} \quad ; \quad q^2 = 4k^2\sin^2\theta/2$$

In limit $m_0 \to 0$ $(r_0 \to \infty)$

$$f^{(1)}(\theta) = \frac{-2mg}{\hbar^2 4k^2 \sin^2\theta/2}$$

$$\frac{d\sigma}{d\Omega} = \left|f^{(1)}(\theta)\right|^2 = \frac{m^2g^2}{4\hbar^2}\;\frac{1}{k^4\sin^4(\theta/2)}$$

For low energy α-particle scattering lab frame is same as CM frame.

$\hbar^2k^2 = 2m_\alpha E$; $g = 2Ze^2$ get Rutherford Scattering,

$$\frac{d\sigma}{d\Omega} = \frac{e^4 Z^2}{4E^2}\sin^{-4}(\theta/2)$$

Note: no $\hbar$ dependence. This is the same as the classical result