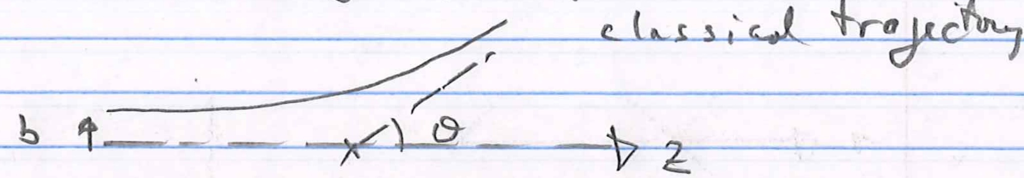
Scattering 2: Partial Waves

Born series (Born approximation) works well at high energies. An alternative approach called the method of partial waves works well at low energies.

Classical scattering impact parameter (b):



annulus between b, $b+db$ scattered into angles $\theta + d\theta$.

$$\frac{d\sigma}{d\Omega}(\text{classical}) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

For example classical hard sphere of radius a, $\sigma = \pi a^2$ cross sectional area.

b is related to angular momentum,

$$L = pb = \hbar k b \qquad \vec{L} \perp \hat{z}$$

For a finite range potential, $V(r) \simeq 0$ for $r > a$ maximum angular momentum is

$$L^2_{max} = \hbar^2 l(l+1) = \hbar^2 k^2 a^2$$

as $k \to 0$ $l = 0$ ("s-wave") dominates

Quantum method of partial wave.
Expand scattering amplitude in terms of spherical harmonics. For $V(|\vec{r}|)$
$f(\theta)$ independent of ϕ and only $Y_{l,0}(\theta)$ contribute.

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0}(\theta)$$ Legendre plolynomials

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1)\, a_l(k)\, P_l(\cos\theta)$$ $(2l+1)$ for convenience

$$P_0(c) = 1; \quad P_1(c) = c, \quad P_2(c) = \tfrac{1}{2}(3c^2-1), \ldots$$

Full wave function solution

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} R_l(r)\, P_l(\cos\theta)$$

where

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial}{\partial r} r\right)^2 R_l + \frac{\hbar^2 l(l+1)}{2\mu r^2} R_l + VR_l = ER_l$$

In limit $kr \gg 1$, $V(r) \to 0$ and we have free particle solution.

Recall, free particle solution are Bessel functions. (See, for example Sakurai)

$$R_l(r) = j_l(kr),\ \eta_l(kr)$$

Since we are interested in $r \to \infty$ solution, we cannot exclude η_ℓ. Asymptotic limits

$$j_\ell(kr) \sim \frac{1}{kr} \sin\left(kr - \frac{\ell\pi}{2}\right)$$

$$\eta_\ell(kr) \sim -\frac{1}{kr} \cos\left(kr - \frac{\ell\pi}{2}\right)$$

Then

$$\psi(r) \underset{r\to\infty}{\longrightarrow} \sum_\ell \left[A_\ell \frac{\sin\theta_\ell}{kr} - B_\ell \frac{\cos\theta_\ell}{kr}\right] P_\ell(\cos\theta)$$

where $\theta_\ell \equiv kr - \ell\pi/2$. We can rewrite as

$$\psi(r) \underset{r\to\infty}{\longrightarrow} \sum_\ell \frac{C_\ell}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) P_\ell(\cos\theta)$$

where δ_ℓ is the ℓ^{th} <u>partial wave phase shift</u>.

ψ includes plane wave,

$$\psi(r) \underset{r\to\infty}{\longrightarrow} e^{ikz} + \underbrace{\frac{e^{ikr}}{r} f(\theta)}_{\psi_{scattered}}$$

To get $f(\theta)$ we must subtract plane wave.
Plane wave expansion,

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr) P_\ell(\cos\theta)$$

$$\underset{kr\gg 1}{\longrightarrow} \sum i^\ell (2\ell+1) \left(\frac{1}{kr}\right) \sin\theta_\ell \, P_\ell(\cos\theta)$$

Coefficients C_ℓ determined by requirement that in the asymptotic region $\psi(r)$ contains only outgoing spherical wave.

$$\psi_{scattered} = \psi - e^{ikz} \xrightarrow[r\to\infty]{}$$

$$\psi_{scatt} \xrightarrow[r\to\infty]{} \sum_\ell \frac{1}{kr} P_\ell(\cos\theta)\Big[C_\ell \sin(\theta_\ell + \delta_\ell) - i^\ell (2\ell+1)\sin\theta_\ell\Big]$$

use $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$, $\theta_\ell \equiv kr - \ell\frac{\pi}{2}$

$$\psi_{scatt} \xrightarrow[r\to\infty]{} \sum_\ell \frac{1}{kr} P_\ell(\cos\theta)\left(\frac{1}{2i}\right)\Big[\ \Big]$$

where

$$[\ ] = C_\ell\left(e^{ikr} e^{-\ell\pi/2} e^{i\delta_\ell} - e^{-ikr} e^{i\frac{\pi\ell}{2}} e^{-i\delta_\ell}\right) - i^\ell(2\ell+1)\left(e^{ikr} e^{-\ell\pi/2} - e^{-ikr} e^{+i\pi\ell/2}\right)$$

incoming wave cancels if

$$C_\ell e^{-i\delta_\ell} = i^\ell (2\ell+1) \quad , \quad i^\ell = e^{i\frac{\pi\ell}{2}}$$

$$\boxed{C_\ell = e^{i\pi\ell/2}(2\ell+1)\, e^{i\delta_\ell}}$$

Then

$$\psi_{sc} \rightarrow \frac{e^{ikr}}{kr} \sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{1}{2i}\right)[\ ] = \frac{e^{ikr}}{r} f(\theta)$$

$$[\ ] = (2l+1)\left(e^{2i\delta_l} - 1\right) = (2l+1)e^{i\delta_l}\sin\delta_l$$

so we find the partial wave scattering amplitude

$$\boxed{f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l}$$

All the scattering physics is in the phase shift δ_l

Total cross section (σ_T)

Orthogonality of Legendre Polynomials,

$$\int d\Omega\, P_l(\cos\theta)P_{l'}(\cos\theta) = \frac{4\pi}{\sqrt{(2l+1)(2l'+1)}}\int d\Omega\, Y_{l0} Y_{l'0}$$

$$= \left(\frac{4\pi}{2l+1}\right)\delta_{ll'}$$

$$\sigma_T = \int d\Omega |f|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

$$\equiv \sum \sigma_l$$ sum of partial wave total cross section

where we have partial-wave unitarity condition (conservation of probability)

$$\sigma_\ell < \frac{4\pi}{k^2}(2\ell+1)$$

Optical Theorem: Forward scattered wave ($\theta=0$) destructively interferes with incident wave.

$$f(0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\underbrace{P_\ell(1)}_{=1}e^{i\delta_\ell}\sin\delta_\ell$$

$$= \frac{1}{k}\sum(2\ell+1)(\cos\delta_\ell + i\sin\delta_\ell)\sin\delta_\ell$$

$$\mathrm{Im}\, f(0) = \frac{1}{k}\sum(2\ell+1)\sin^2\delta_\ell = \frac{k}{4\pi}\sigma_T$$

$$\boxed{\sigma_T = \frac{4\pi}{k}\,\mathrm{Im}\, f(0)}$$

<u>Finding the phase shift δ_ℓ</u>

We have to solve for R_ℓ,

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{d}{dr}r\right)^2 R_\ell + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} R_\ell + V(r) R_\ell = E R_\ell$$

then take the limit $R_\ell(r) \xrightarrow[r\to\infty]{}$

and compare to asymptotic limit of <u>full wave function</u> limit,

$$\Psi(r) \underset{r\to\infty}{\longrightarrow} \sum_{\ell=0}^{\infty} \frac{C_\ell}{kr} \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right) P_\ell(\cos\theta)$$

$$\text{with} \quad C_\ell = e^{i\pi\ell/2}(2\ell+1)e^{i\delta_\ell}$$

This is the hard part!

Simplest example, hard sphere scattering

$$V(r) = \begin{cases} \infty & r<a \\ 0 & r>a \end{cases}$$

For $l=0$, let $U = Rr$ and for $r>a$

$$-\frac{\hbar^2}{2\mu} U'' = EU$$

with boundary condition $U(a)=0$

solutions, $U = \sin, \cos$ or

$$R = A\, j_0(kr) + B\, \eta_0(kr)$$

$$= \frac{1}{kr}\left[A \sin kr - B \cos kr\right]$$

$$R(a) = 0 = \frac{1}{ka}\left[A \sin ka - B \cos ka\right]$$

$$B/A = \tan(ka) \quad \text{so}$$

$$R(r) = \frac{A}{kr}\left[\sin kr - \tan(ka) \cos kr\right]$$

This is $(r>a)$ asymptotic form. Compare

$$\frac{C_0}{kr} \sin(kr+\delta_0) = \frac{C_0}{kr}\left[\sin kr \cos\delta_0 + \cos kr \sin\delta_0\right]$$

$$= \frac{C_0}{kr} \cos\delta_0 \left[\sin kr + \tan\delta_0 \cos kr\right]$$

we find $\boxed{\delta_0 = -ka}$

Total $l=0$ cross section

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2(ka) \xrightarrow[ka \to 0]{} 4\pi a^2$$

which is four times classical hard sphere.

HW # 10 - resonance scattering, Ramsauer-Townsend effect. Near zero scattering, "transmission resonance". Observed for low energy ($E \sim 0.7\,eV$) e-atom scattering by Noble gases.

Scattering length

S-wave phase shift in low-energy limit has simple, physical interpretation.

define scattering length $a = -\lim_{k \to 0} f(\theta)$

$$f(\theta) \xrightarrow[k \to 0]{} \frac{e^{i\delta_0}}{k} \sin\delta_0 = \frac{1}{k}(\cos\delta_0 + i\sin\delta_0)\sin\delta_0$$

k-> 0 limit picks out l=0

for $\delta_0 \ll 1$, $f(\theta) \xrightarrow[k \to 0]{} \frac{\delta_0}{k}$

then $\boxed{a = -\frac{\delta_0}{k}}$

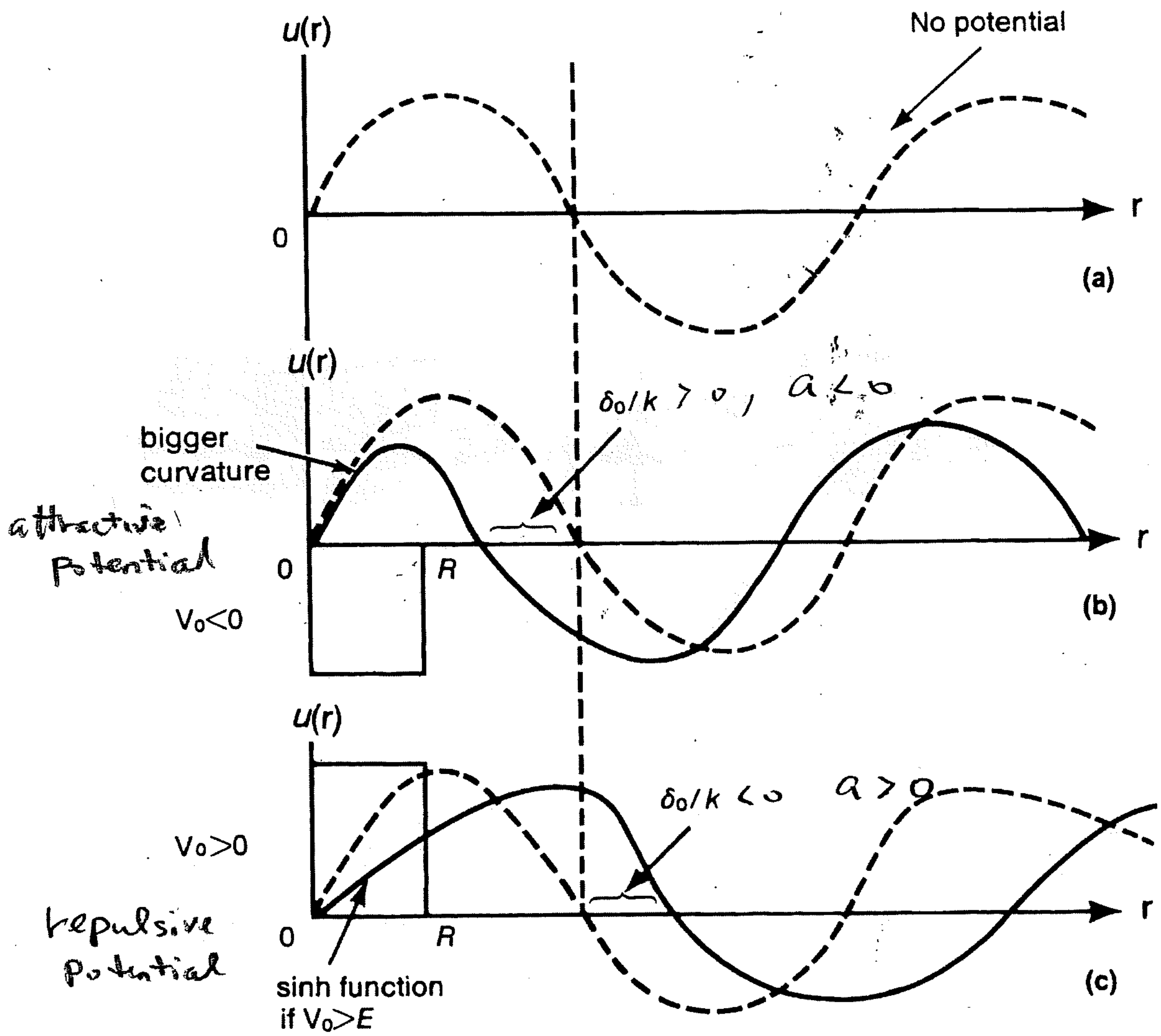


FIGURE 7.8 Plot of $u(r)$ versus r. (a) For $V = 0$ (dashed line). (b) For $V_0 < 0$, $\delta_0 > 0$ with the wave function (solid line) pushed in. (c) For $V_0 > 0$, $\delta_0 < 0$ with the wave function (solid line) pulled out.

Figure from Sakurai

Resonance Scattering

Near short lived ("quasi") bound state
Cross section exhibits an enhancement
called a resonance.

Phase shift is function of k or E etc.

$$\delta_\ell(E) = \pi/2 \quad \text{resonance condition}$$

$$f_\ell = (2\ell+1)\, a_\ell(k) P_\ell(\cos\theta)$$

$$= (2\ell+1) \frac{e^{i\delta_\ell}}{k} \sin\delta_\ell P_\ell(\cos\theta)$$

$$= (2\ell+1) \frac{1}{k} \frac{1}{e^{-i\delta_\ell}} \sin\delta_\ell P_\ell(\cos\theta)$$

$$= (2\ell+1) P_\ell(\cos\theta) \frac{1}{k}\left(\frac{\sin\delta_\ell}{\cos\delta_\ell - i\sin\delta_\ell}\right)$$

$$= (2\ell+1) P_\ell(\cos\theta) \frac{1}{k}\left(\frac{1}{\cot\delta_\ell - i}\right)$$

Taylor expand $\cot\delta_\ell = \underbrace{\cot\delta_\ell(E_0)}_{=0} + (E-E_0) \left.\frac{d\cot\delta_\ell}{dE}\right|_{E_0}$

$$\left.\frac{d\cot\delta_\ell}{dE}\right|_{E_0} = -\frac{1}{\sin^2\delta_\ell} \left.\frac{d\delta_\ell}{dE}\right|_{E_0} = -\left.\frac{d\delta_\ell}{dE}\right|_{E_0}$$

$$\equiv \frac{-2}{\Gamma}$$

then $\cot\delta_\ell \cong -\frac{2}{\Gamma}(E-E_0)$

resonance parameterized by E_0, Γ

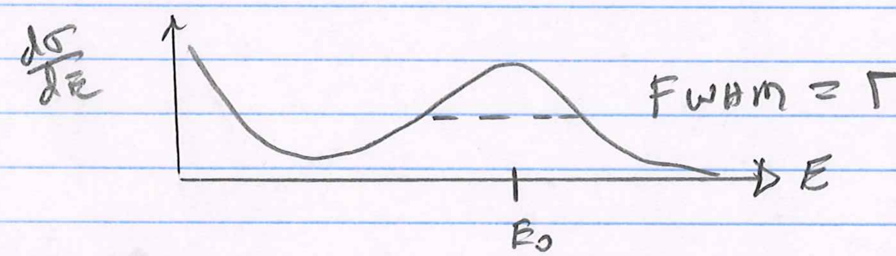
$$a_\ell(k) \simeq -\frac{1}{k}\,\frac{\Gamma/2}{(E-E_0)+\frac{i\Gamma}{2}}$$

with $\int d\Omega\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) = \frac{4\pi}{2\ell+1}\,\delta_{\ell\ell'}$

$$\sigma = \int d\Omega\, |f|^2 = \sum_\ell \sigma_\ell$$

$$\sigma_\ell = \frac{4\pi(2\ell+1)}{k^2}\,\frac{\Gamma^2/4}{(E-E_0)^2+\left(\frac{\Gamma}{2}\right)^2}$$

This shape is called a Breit-Wigner resonance



resonance lifetime $\tau = \frac{\hbar}{\Gamma}$

For example,

Branching ratios:

$$p+\gamma \to \Delta^+(J=3/2) \to \begin{cases} p\,\pi^0 \\ n\,\pi^+ \end{cases} 99.4\%$$

$$p\gamma \quad 0.6\%$$

$m_\Delta = 1236$ MeV, $\Gamma = 115$ MeV

$\sigma(E_0) \simeq 200$ mB (milli barns)

GZK effect

Ultra-high energy cosmic ray protons interact with 2.7°K CMB photons.

$$\langle E_\gamma \rangle_{CMB} \cong 10^{-4}\,eV = 10^{-10}\,MeV$$

some relativistic kinematics:

proton 4-momentum $\hat{P}_p = (E_p, E_p\hat{z})$ @ high energy

photon 4-momentum $\tilde{P}_\gamma = (E_\gamma, -E_\gamma\hat{z})$

$$(\hat{P}_p + \tilde{P}_\gamma)^2 = m_\Delta^2 \qquad \hat{P}_p^2 = m_p^2, \quad \hat{E}_\gamma^2 = 0$$

$$\hat{P}_p \cdot \hat{P}_\gamma = 2E_pE_\gamma \quad \text{then}$$

$$m_p^2 + 2(2E_pE_\gamma) = m_\Delta^2 \qquad m_\Delta = 1232\,MeV$$

then $$E_p = \frac{1}{4}\frac{1}{E_\gamma}(m_\Delta^2 - m_p^2) \cong 10^{15}\,MeV$$
$$= 10^{21}\,eV$$

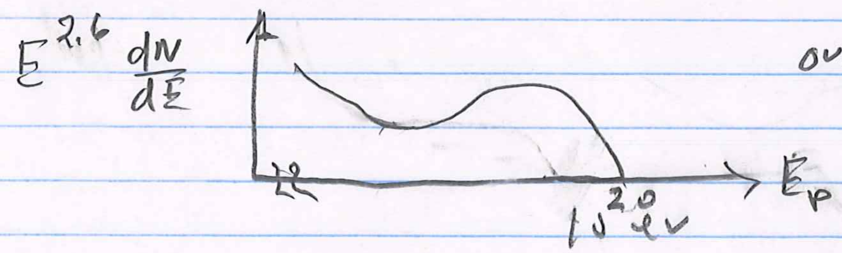
taking into account CMB photon energy distribution,

$$E_p^{th} \sim 5\times10^{19}\,eV$$

Expect cutoff in cosmic ray spectrum for source distance $\gtrsim 50$ MPc

Evidence from HiRes, Auger (power-law spectrum)

Some controversy over cosmic ray composition (p, Fe, ...)



mean free path l=1/nσ where n is the photon density

GZK cutoff in cosmic rays

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Ultra high energy cosmic ray spectrum from the HiRes experiment, Fig. 1. [1] The appearance of the cut-off implies that the sources are $> 50\text{Mpc} = 163\text{MLy}$ distant.

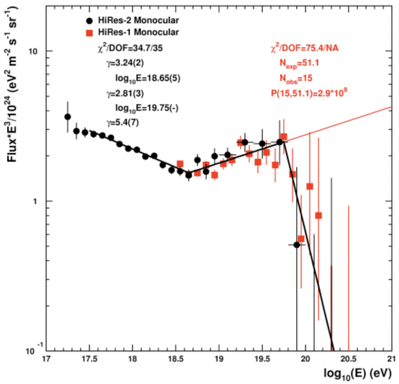

Figure 1: Power law fits to the spectrum and the statistical significance of the GZK cutoff determination

[1]Final Results from the High Resolution Fly's Eye (HiRes) Experiment P. Sokolsky (for the HiRes Collaboration), XVI International Symposium on Very High Energy Cosmic Ray Interactions ISVHECRI 2010, Batavia, IL, USA (28 June-2 July 2010)