Phys 522

A Bit of Special Relativity

In preparation for Dirac equation

Invariant interval

for Euclidean vector $\vec{r}\cdot\vec{r}$ invariant under rotation, uniquely gives rotation's linear transformation. Similarly, ($c=1$ units)

$$t^2 - x^2 = \text{invariant}$$

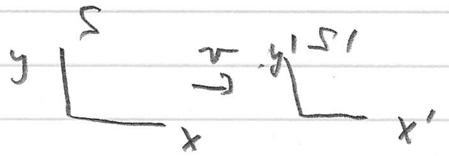
uniquely gives Lorentz boost.

4-vector $x^\mu = (t, \vec{x})$

choose <u>metric</u> $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$

$$x^\mu x_\mu = t^2 - |\vec{x}|^2 \quad \text{Einstein summation convention}$$

Boost



suppress y, z components which are unchanged:

$$\gamma = (1-v^2)^{-1/2} \qquad 1 < \gamma < \infty$$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

define boost angle as $\tanh\theta_x = v_x$ hyperbolic tangent

$$\begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \equiv \bar{B}(\theta)$$

$$\gamma^2 - (\gamma v)^2 = \frac{1}{1-v^2} - \frac{v^2}{1-v^2} = 1$$

$$\cosh^2 - \sinh^2 = 1$$

note on upper, lower indicies.

The position 4-vector $x^\mu = (t, \vec{x})$ is naturally a contravariant vector,

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu \quad \text{transformation rule}$$

transforms opposite to basis, hence "contra".

The gradient is naturally covariant. say acting on a scalar,

$$\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}$$

we write covariant vectors with lower indicies,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

Boosts in same direction add:

$$B_x(\theta_1+\theta_2) = B_x(\theta_1)\,B_x(\theta_2)$$

addition of velocity formula is hyperbolic tangent identity:

$$\tanh(\theta+\theta_r) = \frac{\tanh\theta + \tanh\theta_r}{1+\tanh\theta\tanh\theta_r}$$

Energy and momentum

invariant proper time interval

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2$$

$$\frac{\Delta t}{\Delta\tau} = \frac{\Delta t}{\sqrt{\Delta t^2 - \Delta x^2}} = \frac{1}{\sqrt{1-\left(\frac{\Delta x}{\Delta t}\right)^2}} \xrightarrow[\Delta t \to 0]{} \frac{1}{\sqrt{1-v^2}} = \gamma$$

Construct 4-momentum from x-4vector

$$p^\mu = m\gamma\frac{d}{dt}(x^\mu) = m\gamma\frac{d}{dt}\begin{pmatrix} t \\ x \end{pmatrix} = m\gamma\begin{pmatrix} 1 \\ v \end{pmatrix}$$

$$p^\mu = (E, \vec{p}) \qquad E = m\gamma$$

$$KE = E - m = (\gamma-1)m$$

and $\gamma = E/m$ easy way to get γ

or $(\gamma-1) = KE/m$

gradient: $x^\mu = (t, \vec{x})$ $x_\mu = (t, -\vec{x})$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

4-Laplacian $\square = \partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$

Rotation generators $(c=1, \hbar=1)$

$$\vec{L} = \vec{X} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$$

or as antisymmetric tensor $i,j = 1,2,3$ spatial

$$L^{ij} = -i\left(x^i \nabla^j - x^j \nabla^i\right)$$

$L_3 = L^{12}$ or $L^i = \frac{1}{2}\varepsilon_{ijk} L^{jk}$

generalize to <u>Lorentz group generators</u>

$$J^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \qquad \partial^\mu = \left(\tfrac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$J^{ij} = L^{ij}$ rotation

$K^i \equiv J^{0i}$ boost generators

$[L^i, L^j] = i\varepsilon^{ijk} L^k$ group

$[L^i, K^j] = i\varepsilon^{ijk} K^k$ transform as vector

$[K^i, K^j] = -i\varepsilon^{ijk} L^k$ does not close

boosts do not form group

define $J_\pm^i = L^i \pm i K^i$

then $[J_\pm^i, J_\pm^j] = -i\varepsilon^{ijk} J_\pm^k$

two SU(2) sub-groups

Lorentz group SO(3,1) covering group SL(2,C)

like rotation group SO(3) covering group SU(2)

4-vector representation

$$L^3 = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rotation about z-axis etc.

$$K^1 = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Boost in x-direction

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} X^{\nu}$$

infinitessimal $\Lambda = I - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{k}$

finite $\Lambda = \exp(-i\vec{\theta}\cdot\vec{L} - i\beta\cdot\vec{k})$

$\vec{L}$ are hermitian, $\vec{k}$ antihermitian

Λ not unitary
Lorentz group is not compact

Spinor Representation (2-dim)

$$\vec{L} = \tfrac{1}{2}\vec{\sigma} \quad \vec{K} = \tfrac{-i}{2}\vec{\sigma}$$

then $\Lambda^{1/2} = \exp\left(-i\dfrac{\vec{\theta}\cdot\vec{\sigma}}{2} - \dfrac{\vec{\beta}\cdot\vec{\sigma}}{2}\right)$