Spontaneous Emission

Spontaneous emission: atom interaction with vacuum.

$$|i\rangle = |\psi_i\rangle |0\rangle$$
$$|f\rangle = |\psi_f\rangle |\vec{k}, \lambda\rangle$$

Contribution from $\hat{H}_1$ is

$$\hat{H}_1 = \frac{e}{mc} \hat{\vec{A}}^{+} \cdot \hat{\vec{p}}, \quad \vec{A}^{+} = \frac{c}{\sqrt{V}} \sqrt{\frac{2\pi\hbar}{\omega}} \hat{a}^{+}_{\vec{k},\lambda} \vec{\varepsilon}^{*} e^{-i\vec{k}\cdot\vec{r}}$$

$$k^{-1} = \frac{\hbar c}{E_\gamma} \gtrsim \frac{200\ \text{eV}\cdot\text{nm}}{10\ \text{eV}} = 20\ \text{nm}$$

$$k a_0 = \frac{0.05\ \text{nm}}{20\ \text{nm}} \ll 1$$

<u>dipole approximation</u>: $e^{-i\vec{k}\cdot\vec{r}} \approx 1$

trick to evaluate $\langle \hat{\vec{p}} \rangle$

$$[\hat{H}_0, \hat{r}_i] = \left[\frac{\hat{p}^2}{2m} - \frac{e^2}{r}, \hat{r}_i\right] = \left[\frac{\hat{p}^2}{2m}, \hat{r}_i\right]$$

$$= -i\hbar \hat{p}_i/m$$

$$\hat{\vec{p}} = \frac{i}{\hbar}[\hat{H}_0, \vec{r}]$$

$$\vec{\varepsilon}^{\,*}_{\vec{k},\lambda} \langle \psi_f | \vec{p} | \psi_i \rangle = \frac{i}{\hbar} \vec{\varepsilon}^{\,*} \langle \psi_f | [\hat{H}_0, \vec{r}\,] | \psi_i \rangle$$

$$= \frac{i}{\hbar} E_\gamma \langle \psi_f | \underbrace{\vec{\varepsilon}^{\,*} \cdot \vec{r}}_{\text{electric dipole}} | \psi_i \rangle$$

$$E_\gamma = E_f - E_i$$

$$|\langle f | \hat{H}_1 | i \rangle|^2 = \frac{c^2}{V} \frac{2\pi\hbar^2}{E_\gamma} \frac{e^2 E_\gamma^2}{(\hbar c)^2} |\langle \psi_f | \vec{\varepsilon}^{\,*} \cdot \vec{r} | \psi_i \rangle|^2$$

Rate is

$$\frac{dR}{d\Omega} = \frac{2\pi}{\hbar} \frac{V}{(2\pi)^3} \frac{E_\gamma^2}{(\hbar c)^3} \frac{c^2}{V} \left(\frac{2\pi\hbar^2}{E_\gamma} \right) e^2 \frac{E_\gamma^2}{(\hbar c)^2}$$

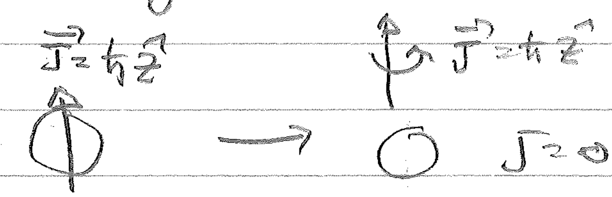
$$\times \sum_\lambda |\langle \psi_f | \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{r} | \psi_i \rangle|^2$$

$$= \frac{1}{2\pi} \frac{\alpha}{c^2 \hbar^3} E_\gamma^3 \sum_\lambda |\langle \psi_f | \vec{\varepsilon}^{\,*} \cdot \vec{r} | \psi_i \rangle|^2$$

check dimensions:

$$\frac{[E]^3 \cdot c}{[E \cdot \ell]^3} [\ell]^2 = c[\ell] = \frac{1}{\text{time}} \quad \checkmark$$

Consider $|2p\rangle = |2,1,1\rangle \rightarrow |1s\rangle$
with right circular polarized photon
emitted along $\hat{z}$ axis :



$$\frac{dR_{+z}}{d\Omega} = \frac{1}{2\pi} \frac{\alpha}{c^2 \hbar^3} E_\gamma^3 \left| \langle 1,0,0 | \vec{\varepsilon}^{\,*} \cdot \vec{r} \, |2,1,1\rangle \right|^2$$

$$\vec{\varepsilon}_\pm = \frac{1}{\sqrt{2}} (\hat{x} \pm i\hat{y})$$ $\vec{\varepsilon}_z$ linear polarization vector

$$\mp \frac{1}{\sqrt{2}} (x \pm iy) = r \sqrt{\frac{4\pi}{3}} \, Y_{1\pm 1}$$

$$z = r \sqrt{\frac{4\pi}{3}} \, Y_{10}$$

$$\vec{r} \cdot \vec{\varepsilon}^{\,*} = r \sqrt{\frac{4\pi}{3}} \left(\varepsilon_-^* Y_{1-1} - \varepsilon_+^* Y_{1+1} + \varepsilon_z z \right)$$

with $Y_{11} = -Y_{1-1}^*$

$$\vec{r} \cdot \vec{\varepsilon}^{\,*} = \left(-\varepsilon_-^* Y_{11}^* + \varepsilon_+^* Y_{1-1}^* + \varepsilon_z z \right)$$

$$= \left(-\varepsilon_+ Y_{11}^* + \varepsilon_- Y_{1-1}^* + \varepsilon_z z \right)$$

$$\langle 1,0,0 | \varepsilon_+ Y_{11}^* | 2,1,1\rangle = (\text{Radial}) \sqrt{\frac{4\pi}{3}} \int d\Omega \, Y_{00} |Y_{11}|^2$$

$$= (\text{radial}) \frac{1}{\sqrt{3}}$$

$$\text{radial} = \int_0^\infty R_{21} R_{10}\, r\,(r^2 dr) = \sqrt{\frac{2}{3}}\ \frac{2^7}{3^4}\, a_0 \equiv RI$$

$$\frac{dR_{+z}}{d\Omega} = \left|\frac{RI}{\sqrt{3}}\right|^2 \frac{1}{2\pi}\ \frac{\alpha}{c^2\hbar^3}\ E_\gamma^3$$

For state $|2,1,1\rangle$ to emit photon in arbitrary direction $\vec{k}$ rotate state to align with direction $\vec{k}$ with $\frac{(1+\cos\theta)}{\sqrt{2}}$

$$\frac{dR_+}{d\Omega} = \left|\frac{RI}{\sqrt{3}}\right|^2 \frac{1}{2\pi}\ \frac{\alpha}{c^2\hbar^3}\ E_\gamma^3\ \frac{(1+\cos\theta)^2}{4}$$

total rate include probability to emit γ of either $\pm$ polarization

$$\frac{dR}{d\Omega} = \frac{dR_+}{d\Omega} + \frac{dR_-}{d\Omega} = \left(\frac{RI}{\sqrt{3}}\right)^2 \frac{1}{2\pi}\ \frac{\alpha}{c^2\hbar^3}\ E_\gamma^3 \left(\frac{1+c^2}{2}\right)$$

$$\int d\Omega \left(\frac{1+c^2}{2}\right) = \left(\frac{8\pi}{3}\right)$$

$$R = \frac{1}{3}\left(\frac{2}{3}\right) \frac{2^{14}}{3^8} \left(\frac{8\pi}{3}\right) \frac{1}{2\pi}\ \frac{\alpha a_0^2 E_\gamma^3}{c^2\hbar^3}$$

$$\boxed{R = \frac{2^{17}}{3^{11}}\ \frac{\alpha a_0^2}{c^2}\ \frac{E_\gamma^3}{\hbar^3}}$$

with $E_\gamma = E_{2p} - E_{1s} = \frac{1}{2} mc^2 \alpha^2 \left(1 - \frac{1}{2^2}\right)$

$$= \frac{3}{8} mc^2 \alpha^2$$

$$\boxed{R_{2p \to 1s} = \left(\frac{2}{3}\right)^8 \alpha^5 \frac{mc^2}{\hbar}}$$

$$\tau = \frac{1}{R} = 1.6 \text{ ns}$$