Phys 492
M. Gold

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Lecture 13: Zeeman Effect

refer to Griffiths p. 244

H-atom in external magnetic field ($\vec{B} = B\hat{e}_z$)

$$\hat{H}' = -\vec{\mu}_e \cdot \vec{B}$$

$$\vec{\mu}_e = -\left(\frac{e\hbar}{2mc}\right)\left(\frac{\vec{L} + g\vec{S}}{\hbar}\right)$$

$g = 2$ point-particle gyromagnetic ratio

$$\mu_B = \frac{e\hbar}{2mc} = 5.79 \times 10^{-9}\ \text{eV/G} \quad \text{Bohr}$$
$$= 5.79 \times 10^{-5}\ \text{eV/T} \quad \text{magneton}$$

$\boxed{\mu_B \propto \text{charge/mass}}$

With $\vec{B} = B\hat{e}_z$ $\quad \hat{H}' = \mu_B B\left(\frac{\hat{L}_z + 2\hat{S}_z}{\hbar}\right)$

Compare $\langle \hat{H}' \rangle$ to $\langle \hat{H}_{SO} \rangle$

$$\langle \hat{H}_{SO} \rangle_{\ell=2} = \frac{1}{96} mc^2 \alpha^4 \begin{cases} 1 & j = 5/2 \\ -2 & j = 1/2 \end{cases}$$

$\langle \hat{H}' \rangle = \langle \hat{H}_{SO} \rangle$ when

$$B \cong \frac{\frac{1}{96} mc^2 \alpha^4}{\mu_B} = \frac{10^{-2}(5 \times 10^5\ \text{eV})\left(\frac{1}{137}\right)^4}{6 \times 10^{-9}\ \text{eV/G}}$$
$$= 5 \times 10^3\ \text{G} = 0.5\ \text{T}$$

Then we have 3 regimes:

$B << T$ weak

$B >> T$ strong

$B \sim T$ intermediate

Weak $\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{r} + \hat{H}_{SO}$

unperturbed states

$$|\phi^0\rangle = |n, l, \tfrac{1}{2}, j, m_j\rangle$$

$$\langle \hat{H}' \rangle = \mu_B B \left[\frac{\langle \hat{J}_z \rangle}{\hbar} + \frac{\langle \hat{S}_z \rangle}{\hbar} \right]$$

where we used $\vec{L} + 2\vec{S} = \vec{J} + \vec{S}$

Since $|\phi^0\rangle$ are not eigenstates of $\hat{S}_z$ we need Clebsch-Gordon coefficients

$$|\underbrace{l \pm \tfrac{1}{2}}_{j}, l, \tfrac{1}{2}, m_j\rangle =$$

$$\left(\frac{l \pm m_j + \frac{1}{2}}{2l+1}\right)^{1/2} |l, m_j - \tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$\pm \left(\frac{l \mp m_j + \frac{1}{2}}{2l+1}\right)^{1/2} |l, m_j + \tfrac{1}{2}\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

then

$$\langle \hat{S}_z \rangle = \frac{1}{2}\left[\frac{l \pm m_j + \frac{1}{2}}{2l+1} - \frac{l \mp m_j + \frac{1}{2}}{2l+1}\right]$$

$$= \frac{\pm m_j}{2l+1} \quad \text{where } \pm \text{ corresponds to } j = l \pm \tfrac{1}{2}$$

adding this to $\langle \hat{J}_z \rangle$ term,

degeneracy removed

$$\langle \hat{H}' \rangle_{j,l,m_j} = \mu_B B\, m_j \underbrace{\left(1 \pm \frac{1}{2l+1}\right)}_{\text{Landé g-factor}}$$

*

Landé g-factor can also be written

In 521 we derived this from WE ~~projection theorem (we.pdf)~~

$$g_L(j,l) = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}$$

for $l=0$ there is no $\pm$ since $j=\frac{1}{2}$ only
$|l-s| \le j \le l+s$. In this case $g_L = 2$

Example $n=2$

$S^{1/2}$ $\langle \hat{H}' \rangle = 2\mu_B B m_j = \pm \mu_B$

$P^{1/2}$ $\left(j = 1 - \frac{1}{2}\right)$

$$\langle \hat{H}' \rangle = \mu_B B m_j \left(1 - \frac{1}{3}\right)$$

$$= \pm \frac{1}{3} \mu_B B$$

$P^{3/2}$ $\left(j = 1 + \frac{1}{2}\right)$

$$\langle \hat{H}' \rangle = \mu_B B m_j \left(1 + \frac{1}{3}\right) = \frac{4}{3} \mu_B B m_j$$

$$m_j = -3/2, -\frac{1}{2}, \frac{1}{2}, 3/2$$

Note – energy splittings $\propto B$.

Strong field

Consider $\hat{H}_{SO}$ as perturbation to

$$\hat{H}_0 = \frac{\vec{p}^{\,2}}{2\mu} - \frac{e^2}{r} + \mu_B B \left(\frac{\hat{L}_z + 2\hat{S}_z}{\hbar} \right)$$

$$|\phi_0\rangle = |E, l, m_l\rangle \, |\tfrac{1}{2}, m_s\rangle$$

$$E_0 = -\frac{1}{2n^2} mc^2 \alpha^2 + \mu_B B (m_l + 2m_s)$$

We don't use $\hat{J}^2, \hat{J}_z$ basis because these are not conserved due to the external torque.

Since there is no longer any degeneracy, we can evaluate $\langle \vec{L} \cdot \vec{S} \rangle$ using non-degenerate perturbation theory. Write

$$\hat{\vec{L}} \cdot \hat{\vec{S}} = \frac{1}{2} \left(\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ \right) + \hat{L}_z \hat{S}_z$$

$$\langle \vec{L} \cdot \vec{S} \rangle = 0 + \langle L_z S_z \rangle = \hbar^2 m_l m_s$$

total fine structure correction is then

$$E'_{fs} = \left(\frac{1}{2n^3} \right) mc^2 \alpha^4 \left\{ \frac{3}{4n} - X \right\}$$

where for $l = 0$, $x = 1$

$$l \neq 0, \quad x = \frac{l(l+1) - m_l m_s}{l(l+\frac{1}{2})(l+1)}$$

See next page

$$E = -\frac{1}{2n^2} mc^2 \alpha^2 + \mu_B B(m_l + 2m_s) + E'_{fs}$$

dependence on B: $\frac{1}{\mu_B}\frac{dE}{dB} = m_l + 2m_s = m_l \pm 1$

l	m_l	$m_l \pm 1$	$n=2$
0	0	1 (-1)	splits into 5
1	1	2 (0)	lines
	0	1 (-1)	
	-1	0 (-2)	

$\uparrow m_s = +\frac{1}{2}$ $m_s = -\frac{1}{2}$

Intermediate Case:

diagonalize $\left[-\vec{\mu}_B \cdot \vec{B}\right] + \left[H_{fs}\right]$ in unperturbed basis

$$|n, l, m_l\rangle |\tfrac{1}{2}, m_s\rangle$$

degeneracy $= 2n^2$

for $n = 2$, 8×8 matrix. label as $L(j, m_j)$,

$P(\frac{3}{2}, \frac{1}{2})$, $P(\frac{1}{2}, \frac{1}{2})$ mix

$P(\frac{3}{2}, -\frac{1}{2})$, $P(\frac{1}{2}, -\frac{1}{2})$ mix

no spin flip

Energy in Strong Zeeman

$$E_k^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left(\frac{3}{4n} - \frac{1}{\ell + \frac{1}{2}} \right)$$

$$E_{so}^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left(\frac{m_\ell m_s}{\ell(\ell+1)(\ell+\frac{1}{2})} \right)$$

$$E_D^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} (1)$$

$$\ell = 0 : \quad E_k^{(1)} + E_D^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left(\frac{3}{4n} - 2 + 1 \right)$$

$$\ell > 0 : \quad E_k^{(1)} + E_{so}^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left(\frac{3}{4n} - \frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1)(\ell+\frac{1}{2})} \right)$$

$$E = -\frac{1}{2} \frac{mc^2 (Z\alpha)^2}{n^2} + \mu_B B (m_\ell + m_s) + E_{fs}^{(1)}$$

$$E_{fs}^{(1)} = \frac{1}{2} \frac{mc^2 (Z\alpha)^4}{n^3} \left(\frac{3}{4n} - X \right)$$

$$\ell = 0 \qquad X = 1$$

$$\ell > 0 \qquad X = \frac{\ell(\ell+1) - m_s m_\ell}{\ell(\ell+1)(\ell+\frac{1}{2})}$$

Zeeman recap   degenerate perturbation theory

$\hat{H}_{Hy} = \frac{\vec{p}^2}{2\mu} - \frac{e^2}{r}$ $|n, \ell, m_\ell, s, m_s\rangle$ $2n^2$ degenerate

$\hat{H}_{fs} = \hat{H}_K + \hat{H}_D + \hat{H}_{SO}$ $|n, j, m_j, \ell, s\rangle$

multiplets $2j+1$ degenerate

$\hat{H}_Z = \mu_B B \left(\frac{\hat{L}_z + 2\hat{S}_z}{\hbar}\right)$ Zeeman

weak $B \ll T$ $\hat{H}_Z$ is perturbation

$\langle \hat{H}_Z \rangle_{n j m_j \ell s} = \mu_B B\, m_j \left(1 \pm \frac{1}{2\ell+1}\right)$ diagonal

where for $\ell = 0$, only + $2j+1$ degeneracy removed

strong $B \gg T$ $\hat{H}_{fs}$ perturbation

$\langle \hat{H}_{fs} \rangle_{n \ell m_\ell s m_s} = \frac{1}{2n^3} mc^2 \alpha^4 \left\{ \frac{3}{4n} - x \right\}$

diagonal

$$x = \begin{cases} 1 & \ell = 0 \\ \dfrac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1)(\ell+\frac{1}{2})} & \ell > 0 \end{cases}$$

for $n=2$ degenerate $8 \rightarrow 5$ lines

intermediate $B \sim T$ diagonalize $\langle \hat{H}_Z + H_{fs} \rangle_{n \ell m_\ell s m_s}$ basis

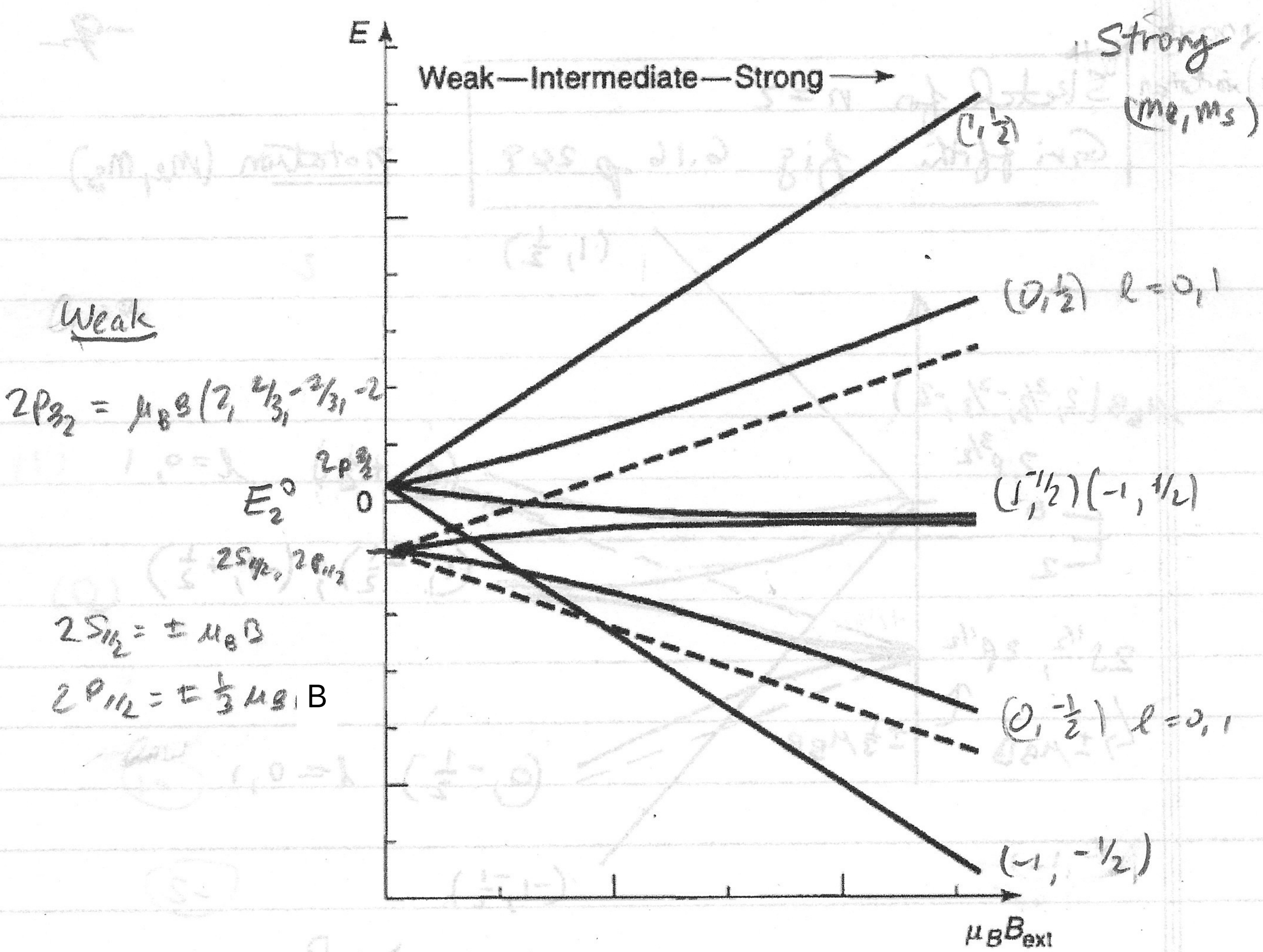

FIGURE 6.12: Zeeman splitting of the $n = 2$ states of hydrogen, in the weak, intermediate, and strong field regimes.

strong splitting: (5 lines)

ℓ	m_ℓ	m_s	$m_\ell \pm 2m_s$
0	0	1/2	1
		-1/2	-1
1	1	1/2	2
		-1/2	0
	0	1/2	1
		-1/2	-1
	-1	1/2	0
		-1/2	-2