Phys 521 Fall 2023

# Semiclassical WKB

Expansion in powers of $\hbar$ (Schiff, Shankar)

$$\psi(x) = \exp\left(\frac{i}{\hbar}\phi(x)\right)$$

Schrödinger time independent

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi$$

$$\psi'' = \frac{-2m(E-V)}{\hbar^2}\psi \equiv -\left(\frac{P(x)}{\hbar}\right)^2\psi$$

$P(x)$ momentum function

$$\psi' = \frac{i\phi'}{\hbar}\psi \quad ; \quad \psi'' = \left(\frac{i\phi'}{\hbar}\right)^2\psi + \frac{i\phi''}{\hbar}\psi$$

equation for $\phi$:

$$-\frac{\phi'^2}{\hbar^2} + \frac{i}{\hbar}\phi' + \frac{p^2}{\hbar^2} = 0$$

expand $\phi = \phi_0 + \hbar\phi_1 + \hbar^2\phi_2 + \cdots$

$$-(\phi_0' + \hbar\phi_1')^2 + i\hbar(\phi_0'' + \hbar\phi_1'') = -p^2$$

$O(\hbar^0)$ $\quad -(\phi_0')^2 = -p^2$

$O(\hbar)$ $\quad -2\phi_0'\phi_1' + i\phi_0'' = 0$

$O(\hbar)$ $\qquad \phi_0' = \pm p(x)$

$$\phi_0 = \pm \int^x p\, dx'$$

$O(\hbar)$ $\qquad \phi_0'' = -i2\phi_0'\phi_1'$

$$\frac{\phi_0''}{\phi_0'} = -2i\phi_1'$$

$$\ln(\phi_0') = -2i\phi_1 + C$$

$$\ln(P) = -2i\phi_1 + C$$

$$\phi_1' = i \ln\sqrt{P} + C'$$

$$\psi(x) = \exp\left(\frac{i}{\hbar}(\phi_0 + \hbar\phi_1)\right)$$

$$= \frac{A}{\sqrt{P}} \exp\left(\pm\frac{i}{\hbar}\int^x P(x')dx'\right)$$

Valid when $O(\hbar) << O(\hbar^0)$ terms

$$\left|\frac{\phi_0''}{\hbar}\right| << \left|\frac{\phi_0'}{\hbar}\right|^2 \qquad \phi_0' = \pm p(x)$$

$$|p'| = \left|p^2 \frac{d}{dx}\frac{1}{p}\right| << \frac{p^2}{\hbar}$$

$$\hbar\left|\frac{d}{dx}\frac{1}{p}\right| << 1$$

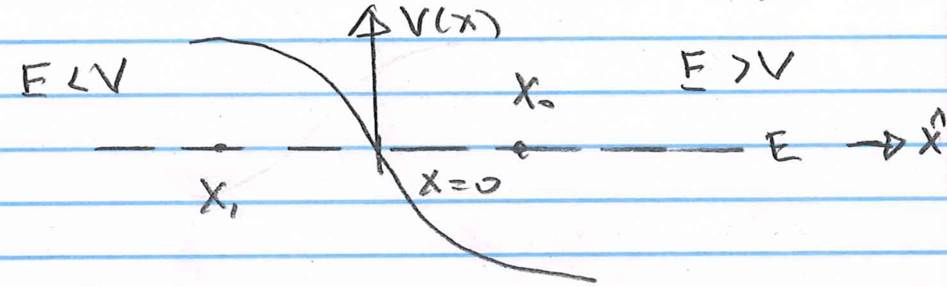
or in terms of de Broglie $\lambda = \frac{\hbar}{p}$

$$\left|\frac{d}{dx}\lambda\right| << 1$$

so valid for slowly varying in x

de Broglie wavelength.

Near classical turning point, WKB is invalid: (following Commins's here)




$$P(x) = 2m(E - V(x))\Big|_{x=0} = 0$$

For $E<V$ $(x<0)$ bound state wave function is non-degenerate and real (up to overall phase).

Proof is in Commins, but for bound state $\vec{j} = \frac{\hbar}{m} \mathrm{Im}(\psi^* \vec{\nabla}\psi) = 0$ implying $\psi$ real.

Then for $x>0$, solution is also non-degenerate and real. Take linear superposition of WKB solutions

$$\psi_+(x_0) = \frac{A}{\sqrt{p}} \sin\left(\frac{1}{\hbar}\int_0^{x_0} p\,dx + \alpha\right)$$

Make linear approximation near $x=0$,

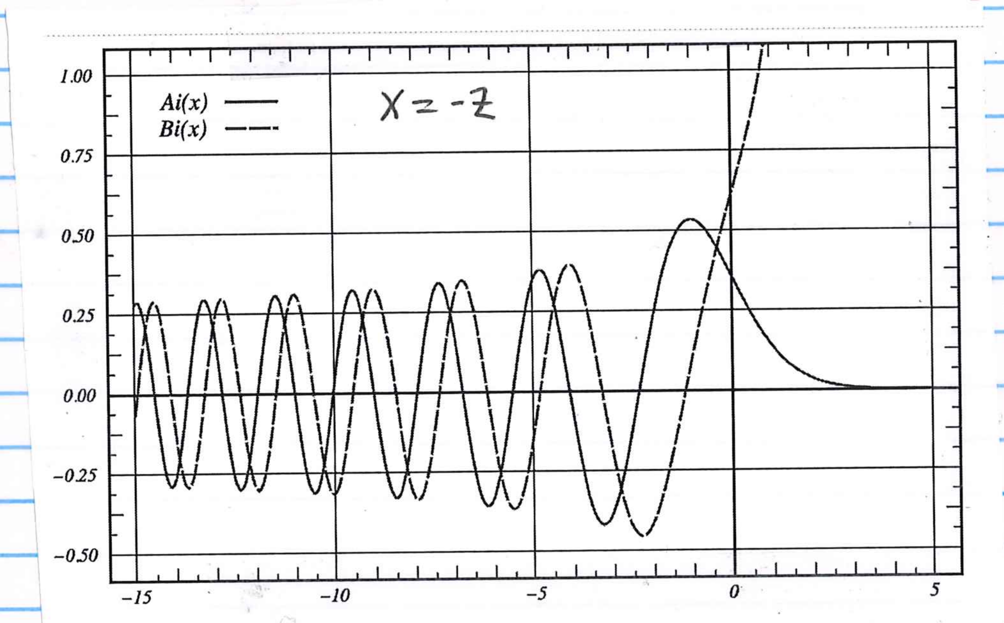
$$V_0(x) = E - F_0 x \quad , \quad F_0 = -\frac{\partial V}{\partial x}\Big|_{x=0}$$

with constant $\beta = \left(\frac{2mF_0}{\hbar^2}\right)^{1/3}$

substitute $z = \beta X$ to get

$$\frac{d^2\psi}{dz^2} + z\psi = 0$$

solution Airy function




note! +z ⟵ | ⟶ −z

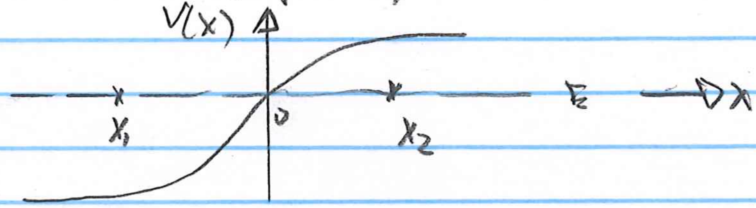
Only need asymptotic:

$$\psi(z) = \frac{\text{const}}{z^{1/4}} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right)$$

then at $x_0$, $\frac{2}{3} z^{3/2} = \frac{1}{\hbar}\sqrt{2m(E - V_0)}$

$$\psi(x_0) = \frac{\text{const}}{\sqrt{p(x_0)}} \sin\left(\frac{1}{\hbar}\int_0^{x_0} p\,dx + \frac{\pi}{4}\right)$$

For other turning point,

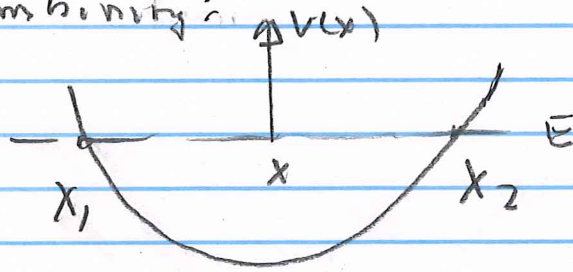



$$\psi(x_1) = \frac{A}{\sqrt{p(x_1)}} \sin\left(\frac{1}{\hbar}\int_{x_1}^{0} p(x)\,dx + \frac{\pi}{4}\right)$$

In classically forbidden region

$$\psi(x_2) = \frac{A}{2\sqrt{|p(x_2)|}} \exp\left(-\frac{1}{\hbar}\int_0^{x_2} |p|\,dx\right)$$

Then combining:




$x_1, x_2$ turning points

for $x_1 < x < x_2$

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p\,dx + \frac{\pi}{4}\right]$$

$$= \frac{A'}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p\,dx + \frac{\pi}{4}\right]$$

with $\int_{x}^{x_2} p\,dx = \int_{x_1}^{x_2} p\,dx - \int_{x_1}^{x} p\,dx$

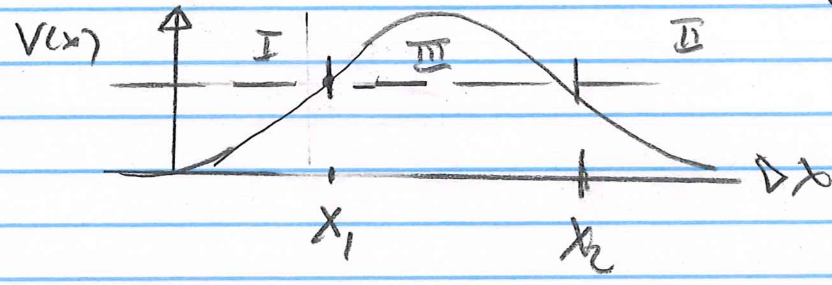
$$\psi(x) = \frac{A'}{\sqrt{p}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x_2} p\,dx - \int_{x_1}^{x} p\,dx + \frac{\pi}{4}\right]$$

the equality then requires:

$$\int_{x_1}^{x_2} p(x)\,dx = \left(m+\frac{1}{2}\right)\pi\hbar \quad ; \quad A = (-1)^m A'$$

Bohr-Sommerfeld quantization rule

Barrier penetration
simplified from Commins (consult for details)




incident wave

$$\psi_{I}(x) = \frac{A}{\sqrt{p}} \sin\left( \frac{1}{\hbar} \int_{x}^{x_1} p(x)\, dx + \frac{\pi}{4} \right)$$

propagated into barrier

$$\psi_{III}(x) = \frac{A}{2\sqrt{|p|}} \exp\left( -\frac{1}{\hbar} \int_{x_1}^{x} |p|\, dx \right)$$

General form of wave heading toward barrier from $x < x_2$,

$$\phi_{III}(x) = \frac{C}{2\sqrt{|p|}} \exp\left( \frac{1}{\hbar} \int_{x}^{x_2} |p|\, dx \right)$$

connects to outgoing wave as

$$\Phi_{II}(x) = \frac{C}{\sqrt{p}} \sin\left( \frac{1}{\hbar} \int_{x_2}^{x} p(x)\, dx + \frac{\pi}{4} \right)$$

transmission is $\quad T = \left| \frac{C}{A} \right|^2$

with $\int_{x_1}^{x} + \int_{x_2}^{x} = \int_{x_1}^{x_2}$

rewrite $\phi_{III}(x)$ as

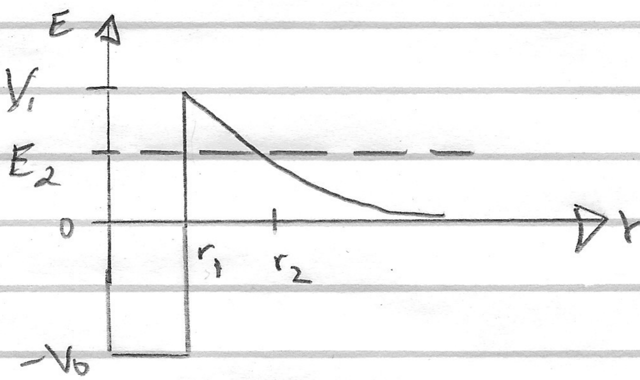
$$\phi_{III}(x) = \frac{C}{\sqrt{|p|}} \exp\left(\frac{1}{\hbar}\int_{x_1}^{x_2} |p(x)| dx\right) \exp\left(-\frac{1}{\hbar}\int_{x_1}^{x} |p(x)| dx\right)$$

$$= \psi_{III}(x)$$

comparing $C \exp\left(\frac{1}{\hbar}\int_{x_1}^{x_2} |p| dx\right) = A$

giving $T = \left|\frac{C}{A}\right|^2 = \exp\left(-\frac{2}{\hbar}\int_{x_1}^{x_2} |p| dx\right)$

# Gamov's Theory of $\alpha$-decay




$\tau = (f T)^{-1}$ $\quad$ $f$ = semi-classical barrier collision frequency

transmission coefficient

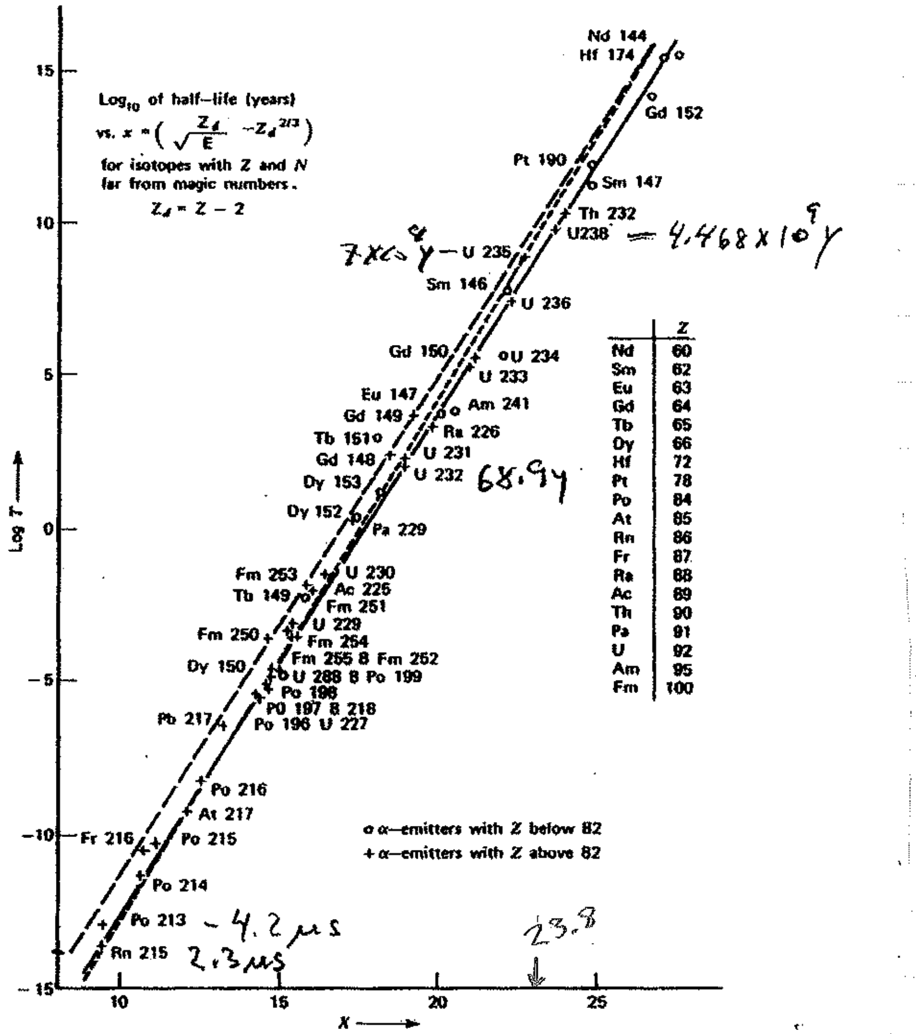
$$T = e^{-2I} \quad \text{where } I = \exp\left(-\frac{1}{\hbar}\int_{r_1}^{r_2} |p(x)|\,dx\right)$$

On homework you will show that $I = K X$ where $X$ depends only on decay energy $E_\alpha$ and daughter nucleus $Z_d = Z - 2$ as

$$X = \frac{Z_d}{\sqrt{E_\alpha[\text{MeV}]}} - Z_d^{2/3}$$

7-6

α-decay data: $\log_{10}$ of ½ life (years) vs
$X = \frac{Z_d}{\sqrt{E}} - Z_d^{2/3}$ where $Z_d = Z-2$ (daughter)




$X_{U238} = 23.8\ \text{MeV}^{-1/2}$

From: Hyde, Perlman and Seaborg
The Nuclear Properties of Heavy Elements, Vol. I
Prentice-Hall, Englewood Cliffs, NJ (1964)

From PDG:

| | $T_{1/2}$ |
|---|---|
| $U^{238}$ | 4.47 Gy |
| $Po^{212}$ | 0.299 μs $= 3\times10^{-7}$ s |