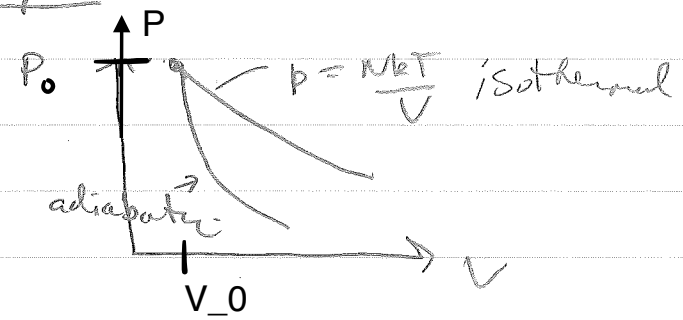
Phys 301

Extra Lecture

Example Adiabatic versus Isothermal

Baierlein



a) Condition for same slope

$$PV^\gamma = P_0 V_0^\gamma \qquad \gamma = \frac{2}{f} + 1$$

isothermal $\left(\frac{dP}{dV}\right)_T = -\frac{NkT}{V^2} = -\frac{P_0 V_0}{V^2}$

adiabatic $\left(\frac{dP}{dV}\right)_{ad.} = \frac{d}{dV}\frac{P_0 V_0^\gamma}{V^\gamma} = \frac{-\gamma}{V}\frac{P_0 V_0^\gamma}{V^\gamma} = \frac{-\gamma}{V} P$

$$\text{ratio} = 1 = \frac{-P_0 V_0}{V^2}\left(\frac{V}{-\gamma P}\right) = \frac{1}{\gamma}\frac{P_0 V_0}{VP}$$

$$= \frac{1}{\gamma}\left(\frac{V_0}{V}\right)^{\gamma-1}$$

$$\frac{V}{V_0} = \exp\left[\left(\frac{1}{\gamma-1}\right)\ln\gamma\right] > 1$$

= 2.3 for nitrogen (f=5)

b) Curves never cross again —

$$\frac{P_{ad}}{P_{iso}} = \left(\frac{V_0}{V}\right)^{\gamma-1} \qquad \gamma > 1$$

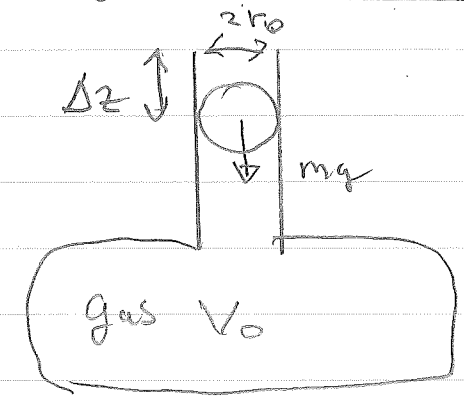
$\frac{V_0}{V} < 1$ so $P_{ad} < P_{iso}$

Ex 3

Baierlein

measuring γ Ruchardt's exp.

distance down from top of tube



precision tube and metal ball

metal ball fits tightly so no gas to leak past sphere
ball compresses gas adiabatically

ball compresses gas until equilibrium

bouyancy $\Delta p \cdot \pi r_0^2 = mg$ gravity

$$\Delta p = \left(\frac{\partial P}{\partial V}\right)_{\substack{\text{adiabatic} \\ \text{quasi-static}}} \Delta V \qquad \Delta V = -\pi r_0^2 \Delta z$$

$$PV^\gamma = P_0 V_0^\gamma \qquad \left(\frac{\partial P}{\partial V}\right) = -\gamma V^{-\gamma-1} P_0 V_0^\gamma$$

$$P = \frac{P_0 V_0^\gamma}{V^\gamma} \qquad = -\frac{\gamma}{V}\left(\frac{1}{V^\gamma}\right) P_0 V_0^\gamma$$

$$\left(\frac{\partial P}{\partial V}\right) = -\gamma\left(\frac{P}{V}\right)$$

$$\Delta p = -\gamma\left(\frac{P}{V}\right)\underbrace{\Delta V}_{-\pi r_0^2 \Delta z} = \frac{mg}{\pi r_0^2}$$

$$\gamma\left(\frac{P}{V}\right)\Delta z = mg\left(\frac{1}{\pi r_0^2}\right)^2 \qquad m = \rho\frac{4\pi}{3} r_0^3$$

$$= \rho\frac{4\pi}{3} r_0^3 g\left(\frac{1}{\pi^2 r_0^4}\right) = \frac{4}{3}\frac{\rho g}{\pi r_0}$$

solve for ΔZ

Ex 4

$$\Delta z = \frac{4\rho}{3}\,\frac{g}{\pi r_0}\left(\frac{1}{\gamma}\right)\frac{1}{(P/V)} \approx P_0/V_0$$

for steel ball $\rho = 7.8\ kg/m^3$

$V_0 = 10\ \ell$ $\quad$ P = atmospheric

$r_0 = 0.8\ cm$

$$\gamma = \frac{f+2}{f} = 5/3 = 1.7$$

$$\frac{7}{5} = 1.4$$

$\gamma = 1.4$, $\quad \Delta z \approx 30\ cm$

$\gamma = 1.7$, $\quad \Delta z = 24\ cm$

<u>Oscillations about equilibrium</u>

$$F_{net} = \pi r_0^2\,\Delta p = -\gamma\left(\frac{P}{V}\right)\underbrace{\Delta V}_{\text{displacement } z(\pi r_0^2)}\ \pi r_0^2$$

z measured from equilibrium

$$m\ddot{z} = -\gamma\left(\frac{P}{V}\right)(\pi r_0^2)^2\, z$$

$$m = \rho\,\tfrac{4}{3}\pi r_0^3$$

$$\ddot{z} = -\gamma\left(\frac{P}{V}\right)\left(\frac{\pi^2 r_0^4}{\frac{4}{3}\pi r_0^3\rho}\right) z$$

frequency of harmonic oscillations

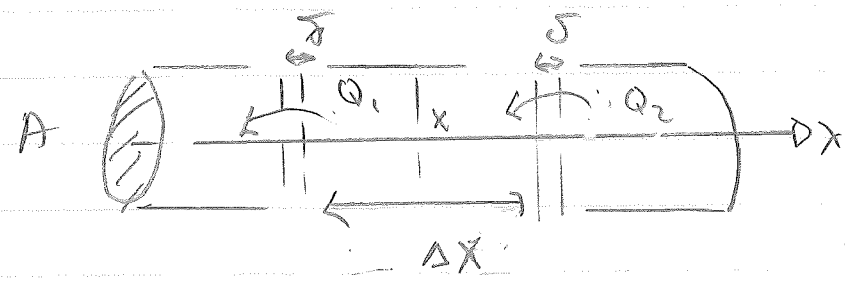
$$\frac{3\pi r_0}{4\rho}$$

$$\boxed{2\pi f = \sqrt{\gamma\left(\frac{P}{V}\right)\frac{3\pi r_0}{4\rho}}}$$

$\gamma = 1.4$, $\quad f = 1\ Hz$

Heat Diffusion (p1.62)

recall at equilibrium $\frac{Q}{\Delta t} = -kA \frac{\Delta T}{\Delta x}$



uniform Bar

$T(x,t)$

$$\frac{Q_1}{\Delta t} = -kA \frac{\Delta T}{\delta} \xrightarrow[\delta \to 0]{\text{limit}} -kA \left.\frac{\partial T}{\partial x}\right|_{x-\Delta x/2}$$

$$\frac{Q_2}{\Delta t} = +kA \frac{\Delta T}{\delta} \xrightarrow[\delta \to 0]{} +kA \left.\frac{\partial T}{\partial x}\right|_{x+\Delta x/2}$$

$$\frac{Q_1+Q_2}{\Delta t} = kA \left[\left.\frac{\partial T}{\partial x}\right|_{x+\frac{\Delta x}{2}} - \left.\frac{\partial T}{\partial x}\right|_{x-\frac{\Delta x}{2}} \right]$$

$$= kA \frac{\partial^2 T}{\partial x^2} \Delta x$$

$$\frac{Q_1+Q_2}{\Delta t} = \frac{\Delta U}{\Delta t} \underset{\Delta t \to 0}{\lim} = \frac{\partial U}{\partial T}\frac{\partial T}{\partial t} = mc \frac{\partial T}{\partial t}$$

$$\frac{m}{A\Delta x} = \rho$$

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$\boxed{\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \qquad K = \left(\frac{k}{\rho c}\right)}$$

Separation of variables

$$\dot{T} = k T''$$

let $T = f(t)\, g(x)$

$$\frac{1}{k}\frac{\dot{f}}{f} = \frac{g''}{g} = -\alpha^2 \quad \text{separation}$$

$$\dot{f} = -\alpha^2 k f$$

$$f(t) = e^{-\alpha^2 k t}$$

exponentially approaches Const. in time

$$g'' + \alpha^2 g = 0$$

boundary conditions determine α

Solution for initial condition of a point source at x=0,t=0

https://en.wikipedia.org/wiki/Heat_kernel#/media/File:Fundamental_solution_to_the_heat_equation.gif

Ex 9

Schroeder asks us to show that this is a solution.

trial solution

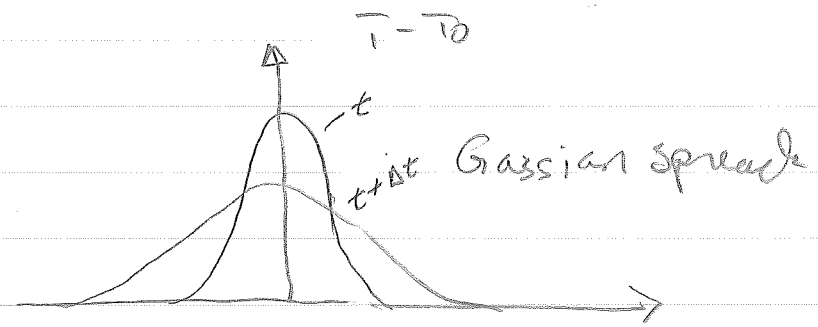
$$T(x,t) = T_0 + \frac{A}{\sqrt{t}} e^{-y} \qquad y = \frac{x^2}{4kt}$$

$$\dot{T} = -\frac{1}{2}\frac{A}{t^{3/2}} e^{-y} + \frac{A}{\sqrt{t}} e^{-y} \frac{x^2}{4kt^2}$$

$$T' = \frac{\partial T}{\partial y}\frac{\partial y}{\partial x} \; ; \quad T'' = \frac{\partial^2 T}{\partial y^2}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{\partial T}{\partial y}\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial y}{\partial x} = \frac{x}{2kt} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{2kt}$$

$$T'' = \frac{A}{\sqrt{t}} e^{-y} \frac{x^2}{4k^2t^2} + \left(-\frac{A}{\sqrt{t}} e^{-y}\right)\left(\frac{1}{2kt}\right)$$



$$\frac{x^2}{2\sigma^2} = \frac{x^2}{4kt} \qquad \boxed{\sigma = \sqrt{2kt}}$$

$$t \to \infty \qquad T(x,t) = T_0$$

Random Walk (1D)

N steps of length ℓ

p = prob to step right $\qquad (p+q)=1$

q = prob to step left

after N steps, m to right, N-m left

$$x = \ell[m-(N-m)] = \ell(2m-N)$$

What is mean $\langle x\rangle$?

Binomial Probability

$$(p+q)^N = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} \equiv \sum_{m=0}^{N} W(m)$$

$$\binom{N}{m} = N!/m!(N-m)!$$

$$\langle m\rangle = \sum_{m=0}^{N} m W(m)$$

trick $\quad \frac{\partial}{\partial p}(p+q)^N = \frac{\partial}{\partial p}(1) = 0$

$$0 = \sum_{m=0}^{N} \binom{N}{m}\left[m p^{m-1}(1-p)^{N-m} - (N-m)p^m(1-p)^{N-m-1}\right]$$

$$= \frac{1}{p}\langle m\rangle - \left(\frac{1}{1-p}\right)\langle N-m\rangle$$

$N-\langle m\rangle$ $\quad$ multiply by $p(1-p)$

$$0 = (1-p)\langle m\rangle - p(N-\langle m\rangle)$$

$$\boxed{\langle m\rangle = Np}$$

So $\langle x \rangle = \ell \left[2\langle m \rangle - N \right] = \ell N (2p - 1)$

Variance $(\Delta m)^2 = \langle m^2 \rangle - \langle m \rangle^2$

Same trick $\langle m^2 \rangle = Npq + N^2 p^2$

$(\Delta m)^2 = \langle m^2 \rangle - \langle m \rangle^2 = Npq$ $(\Delta p)^2 = pq/N$

ratio $\dfrac{\Delta m}{\langle m \rangle} = \sqrt{\dfrac{q}{p}} \dfrac{1}{\sqrt{N}} \underset{p=q=\frac{1}{2}}{=} \dfrac{1}{\sqrt{N}}$

$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle = \ell^2 \langle (2m - N - N(2p-1))^2 \rangle$

$= \ell^2 \langle (2m - 2Np)^2 \rangle$

$= \ell^2 \langle 4m^2 - 8Nmp + 4N^2p^2 \rangle$

$= \ell^2 \left[4\langle m^2 \rangle - 8Np\langle m \rangle + 4N^2p^2 \right]$

$= \ell^2 \left[4Npq + 4N^2p^2 - 8N^2p^2 + 4N^2p^2 \right]$

$(\Delta x)^2 = \ell^2 4Npq \underset{p=q=\frac{1}{2}}{=} \ell^2 N$

$\boxed{\Delta x = \ell\sqrt{N}}$ $p = q = \frac{1}{2}$

In large N, m limit, Binomial becomes Gaussian. For $p=q=\frac{1}{2}$, $\langle x \rangle = 0$ and probability,

$$\tilde{P}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right\}$$

$$\sigma = \ell\sqrt{N}$$

Measurement error – best estimates $\tilde{p}, \tilde{q}$ are

$$\tilde{p} \cong \frac{m}{N}, \quad \tilde{q} \cong \left(\frac{N-m}{N}\right)$$

$(\Delta p)^2 = pq/N$

error on $\tilde{p}$, $\Delta\tilde{p} = \frac{1}{\sqrt{N}}\sqrt{\left(\frac{m}{N}\right)\left(\frac{N-m}{N}\right)}$

$$= \sqrt{\frac{(m)(N-m)}{N^3}}$$

Back to diffusion in 3 dimensions

for diffusion with mean speed $\langle v \rangle$ and mean free path ℓ (recall $\ell \pi d^2 n = 1$)

n = density

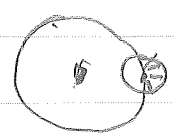
molecular diameter d

$$N = \frac{\langle v \rangle t}{\ell}$$ time t

$$\Delta X = \sqrt{\langle v \rangle \ell t}$$

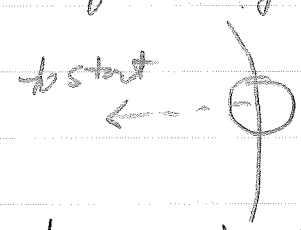
Baierlein has nice argument for $\sqrt{t}$ dependence

small t



more likely to diffuse out

after long time



almost equally likely to diffuse in or out

Brownian motion: dance of pollen grains observed by botanist Robert Brown in 1827

micron size particles suspended in liquid

you can find translation of Einsteins paper,

Ex - 14

http://www.maths.usyd.edu.au/u/UG/SM/MATH3075/r/Einstein_1905.pdf

Einstein Brownian motion

One of three 1905 papers

$$(\Delta x)^2 = \left(\frac{RT}{3\pi N_A r \eta}\right) t$$ displacement in direction x

r radius of particle

η viscous drag

displacement probability given by diffusion

$$\frac{\partial \tilde{p}}{\partial t} = D \frac{\partial^2 \tilde{p}}{\partial x^2}$$

$$D = \left(\frac{RT}{N_A}\right)\frac{1}{6\pi \eta r} \qquad (\Delta x)^2 = 2Dt$$

Perrin measured N_A using particle from gum extract disolved in water, "gamboge"

$N_A = 7.15 \times 10^{+23}$

repeat Newburgh, Peidle, Rueckner AJP 2006

$2r\ (\mu m)$	$N_A / 10^{23}$
0.50 ± 0.02	8.2 ± 0.4
1.09 ± 0.04	6.4 ± 0.3
2.06 ± 0.02	5.7 ± 0.2

by counting atoms in ^{28}Si crystal ArXiv:1010.237 (2010)

$N_A = 6.022\ 1404\ (18) \times 10^{23}$

2016 https://arxiv.org/ftp/arxiv/papers/1601/1601.01450.pdf