Lecture #11 Calculus of variations II

"δ-notation" path $y(t)$, extremitize

$$S(\alpha) = \int_{t_i}^{t_f} f(y, \dot{y}; t)\, dt$$

Varied paths $y(t) = y_0(t) + \alpha\,\eta(t)$
where $\eta(t_i) = \eta(t_f) = 0$

$$\frac{\partial S}{\partial \alpha} d\alpha = \int_{t_i}^{t_f} \left(\frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial f}{\partial \dot{y}} \right) \eta(t)\, d\alpha\, dt$$

define variational $\delta y \equiv \eta(t)\, d\alpha$

$$\delta S \equiv \frac{\partial S}{\partial \alpha} d\alpha$$

then

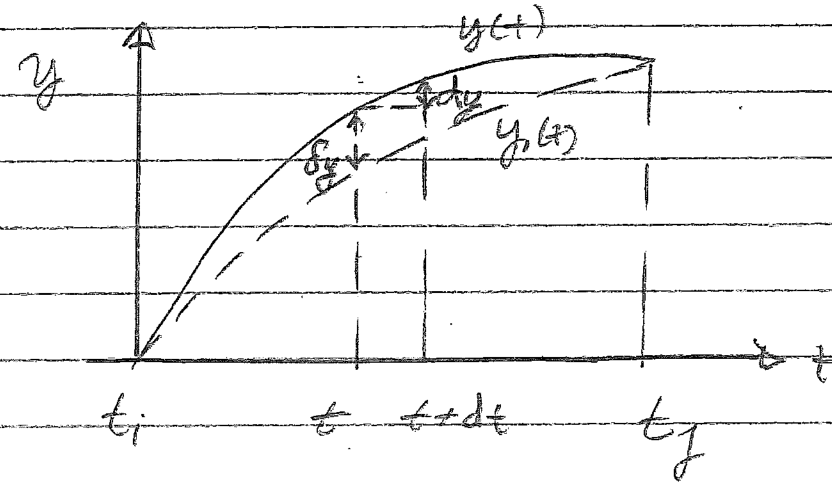
$$\delta S = \delta \int f(y, \dot{y}; t)\, dt =$$

$$\int \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial \dot{y}} \left(\frac{d}{dt} \delta y \right) \right] dt$$

$$= \int \left(\frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial f}{\partial \dot{y}} \right) \delta y\, dt$$

Formally use δy just like ordinary derivative.

Infinitessimal dy Versus δy



$dy = y(t+dt) - y(t) = \left(\frac{dy}{dt}\right) dt$ compares $y(t)$ at different times

δy compares $y(t)$ to $y_0(t)$ at same time.

In what sense is δy infinitessimal?

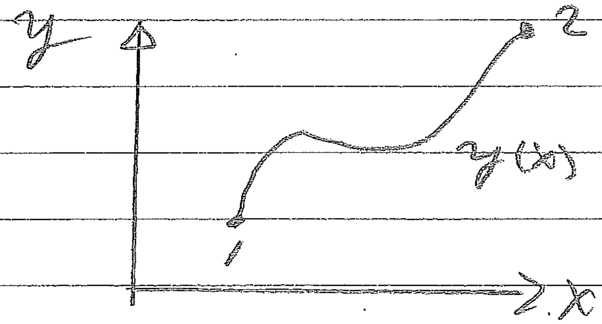
Classically, we do not know and do not care since all we need is Euler-Lagrange. We saw that if $f = \frac{1}{2} m \dot{y}^2$ (kinetic energy) then S has units of energy-time

In quantum mechanics $\hbar = \frac{h}{2\pi}$ has units of energy · time.

In Q.M. we can compare δS to $\hbar$.
This is the basis for Feynman's formulation of Q.M.! (Feynman's PhD thesis!!)

Some more examples

Example: Shortest distance between 2 points on plane



$$ds^2 = dx^2 + dy^2$$

$$\text{path length } S = \int_1^2 ds = \int_1^2 \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$$

$$\therefore f = \sqrt{(y')^2 + 1}$$

$$\frac{\partial f}{\partial y} = 0 = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

$\frac{\partial f}{\partial y'} = k$ where k is a constant

$\frac{y'}{\sqrt{y'^2+1}} = k$ solve for y' –

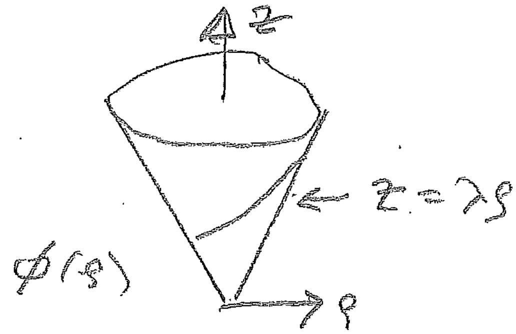
$\frac{dy}{dx} = y' = \frac{k}{\sqrt{1-k^2}} = \tan\theta$

integrate to get $\boxed{y(x) = x\tan\theta + y_0}$

Now go back and calculate the distance from the integral and get the distance formula of Euclidean Geometry.

Geodesic on a Cone

Taylor, problem 6.17



$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 = (1+\rho^2\phi'^2+\lambda^2)\, d\rho^2$$

$$ds = (1+\rho^2\phi'^2+\lambda^2)^{1/2} d\rho = f(\phi, \phi', \rho)\, d\rho$$

$$\frac{\partial f}{\partial \phi} = 0 = \frac{d}{d\rho}\left(\frac{\rho^2 \phi'}{f}\right)$$

$$\rho^2 \phi' = c f = c(1+\rho^2\phi'^2+\lambda^2)^{1/2}$$

$$(\rho^4 - c^2\rho^2)\phi'^2 = c^2(1+\lambda^2)$$

$$\phi' = \frac{c\sqrt{1+\lambda^2}}{\rho\sqrt{\rho^2-c^2}} = \frac{\frac{1}{c}\sqrt{1+\lambda^2}}{\frac{\rho}{c}\sqrt{\left(\frac{\rho}{c}\right)^2-1}}$$

$$\phi(\rho) = \sqrt{1+\lambda^2}\,\cos^{-1}\left(\frac{c}{\rho}\right) + \phi_0$$

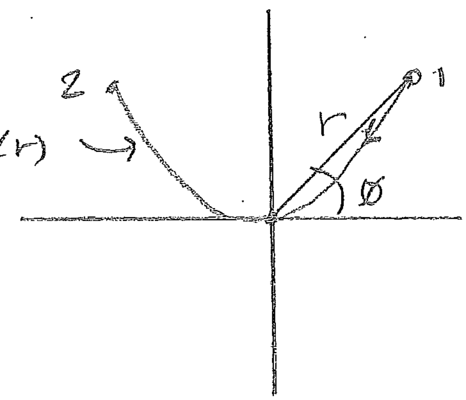
$$\boxed{\rho = \frac{\rho_0}{\cos\left(\frac{\phi-\phi_0}{\sqrt{1+\lambda^2}}\right)}}$$

where the constant c is now called rho-zero

"Attractive" refractive index

Taylor, problem 6.24

see gradient-index GRIN optics



$n = \frac{a}{r^2}$

a is a constant with dim [length^2]

$v = \frac{c}{n} = \frac{c}{a} r^2$

c is the speed of light

Fermat's principle: minimize $\int \frac{ds}{v}$

$ds = (1 + r^2 \phi'^2)^{1/2}\, dr$

$f(\phi', r) = \frac{\sqrt{1 + r^2\phi'^2}}{\frac{c}{a} r^2}$

$\frac{\partial f}{\partial \phi'} = \frac{r^2 \phi'}{\sqrt{1 + r^2\phi'^2}} \frac{a}{c r^2} = \text{constant} = \frac{a}{c}\frac{1}{r_0}$

$r_0^2 \phi'^2 = 1 + r^2 \phi'^2$

$r_0^2 \left(1 - \frac{r}{r_0}\right)^2 \phi'^2 = 1$

$\frac{d\phi}{dr} = \frac{1}{r_0} \frac{1}{\sqrt{1 - (r/r_0)^2}}$

$\phi = \sin^{-1}\left(\frac{r}{r_0}\right)$

$\boxed{r = r_0 \sin\phi}$

Circle through origin

<u>Multiple dimensions:</u>

$x(t), y(t) \qquad f(x, \dot{x}, y, \dot{y}; t)$

vary paths x, y independently

subject to $\delta x(t_1) = 0 = \delta x(t_2)$ & $\delta y(t_1) = 0 = \delta y(t_2)$

$$\delta S = \delta \int_1^2 f(x, \dot{x}, y, \dot{y}; t)$$

$$= \int_1^2 \left[\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \right) \delta x \, dt + \left(\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} \right) \delta y \, dt \right]$$

giving two Euler-Lagrange equations.

<u>Simple example:</u> shortest distance in plane

$$\text{length} = \int \sqrt{dx^2 + dy^2} = \int \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$$

two Euler equations are

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0$$

$$\dot{x} = c_x \sqrt{\dot{x}^2 + \dot{y}^2} = c_x v$$

$$\dot{y} = c_y \sqrt{\dot{x}^2 + \dot{y}^2} = c_y v$$

$$\dot{x}^2 + \dot{y}^2 = v^2 = (c_x^2 + c_y^2) \, v^2$$

so $c_x^2 + c_y^2 = 1$ and we can write $\frac{c_y}{c_x} = \tan\theta$

$$\frac{\dot y}{\dot x} = \frac{dy}{dx} = \tan\theta$$

$y = \tan\theta \, x + y_0$ equation for line

$$\left.\begin{aligned} x(t) &= \cos\theta \, v t \\ y(t) &= \sin\theta \, v t + y_0 \end{aligned}\right\}$$ parameterized by t and v with $t_1 \equiv 0$

Constraints – Lagrange multipliers

Suppose $x(t), y(t)$ are not independent but subject to a constraint that can be written as

$$g(x, y, t) = 0$$

e.g. rolling w/o slipping $x - r\phi = 0$

not an inequality, does not depend on $\dot x$ or $\dot y$.

then $\delta x, \delta y$ are not independent but are related by

$$\delta g = \frac{\partial g}{\partial y}\delta y + \frac{\partial g}{\partial x}\delta x = 0$$

$$\delta x = \left(-\frac{\partial g/\partial y}{\partial g/\partial x}\right)\delta y$$

Variation becomes

$$\delta S = \int_1^2 \left\{ \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot x}\right)\underbrace{\left(\frac{-\partial g/\partial y}{\partial g/\partial x}\right)}_{\delta x \text{ term}} + \left(\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot y}\right)\right\}\delta y \, dt$$

$$= \int_1^2 \left\{ \underbrace{\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot x}\right)\frac{1}{\partial g/\partial x}}_{= -\lambda(t)} + \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot y}\right)\frac{-1}{\partial g/\partial y}}_{= \lambda(t)}\right\} \delta y \left(-\frac{\partial g}{\partial y}\right) dt$$

Requirement that $\delta J = 0$ implies

$$\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right)\left(\frac{1}{\partial g/\partial x}\right) = -\lambda(t)$$

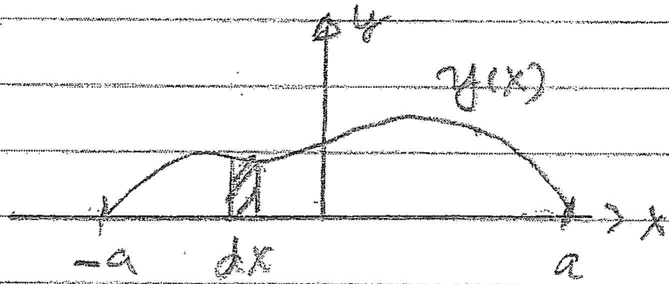
$$\left(\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\right)\left(\frac{-1}{\partial g/\partial y}\right) = \lambda(t)$$

Where $\lambda(t)$ is unknown function of t called Lagrange multiplier.
In context of constrained motion can be used to get force of constraint.

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} + \lambda \frac{\partial g}{\partial x} = 0 \quad \text{etc.}$$

Example - Isoperimetric problem

Queen Dido, Carthage 900 B.C. largest area enclosed by fixed length.



Area $A = \int_{-a}^{a} y(x)\,dx$ maximize — integral, or functional

length $L = \int ds = \int_{-a}^{a} (1 + y'^2)^{1/2}\,dx$ constraint

We can incorporate the constraint in this case by a constant Lagrange multiplier
λ (dim of length) see Gelfand, Fomin p. 43

$$S = A + \lambda L = \int_{-a}^{a} \left(y + \lambda \sqrt{(1 + y'^2)} \right) dx$$

In variation we must treat $A(y, y'; x)$ and $L(y, y'; x)$ as functionals —

$$\left(\frac{\partial A}{\partial y} - \frac{d}{dx}\frac{\partial A}{\partial y'} \right) + \lambda \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} \right) = 0$$

$$\frac{\partial A}{\partial y} = 1 \qquad \frac{\partial A}{\partial y'} = 0 \qquad \frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

giving $\;1 - \lambda \dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1+y'^2}}\right) = 0$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{y'^2+1}}\right) = \frac{1}{\lambda}$$

integrate $\;\dfrac{y'}{\sqrt{y'^2+1}} = \dfrac{x + c_1}{\lambda}$

re-arrange $\;\dfrac{dy}{dx} = \dfrac{\pm(x - c_1)}{\left[\lambda^2 - (x - c_1)^2\right]^{1/2}}$

integrate $\;y = \mp\sqrt{\lambda^2 - (x - c_1)^2} + c_2$

$$(y - c_2)^2 + (x - c_1)^2 = \lambda^2$$

radius of circle is λ.

with our initial conditions $y = 0$ at $x = a$, $x = -a$

$x^2 + y^2 = a^2$. now ~~let~~ $y = a\sin\theta$, $x = a\cos\theta$

$$L = \int_{-a}^{a} \sqrt{1 + y'^2}\,dx = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \pi a$$

Invariance of Euler-Lagrange (Gelfand, Fomin)

$$S[y] = \int f(y, y'; x)\, dx$$

[ ] ≡ functional of path $y(x)$ and $y' \equiv \frac{dy}{dx}$

Change variables $y(x) \to v(u)$

where $x = x(u,v)$ and $y = y(u,v)$

Jacobian $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$

$$\frac{dx}{du} = \left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}\frac{dv}{du}\right)$$

$$\frac{dy}{du} = \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\frac{\partial v}{du}\right) \qquad \text{let } v' \equiv \frac{dv}{du}$$

$$\frac{dy}{dx} = \left(\frac{dy/du}{dx/du}\right) = y'(u, v, v')$$

gives

$$\acute{S}[v] = \int \underbrace{f(y; y', x)\left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} v'\right)}_{\equiv f_u(v, v'; u)} du$$

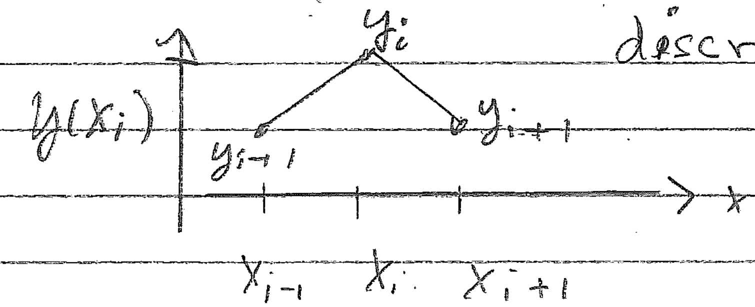
where $y(u,v)$; $y'(u,v,v')$; $x(u,v)$ then

$$\frac{\partial f_u}{\partial v} - \frac{d}{du}\frac{\partial f_u}{\partial v'} = 0 \qquad \text{transformed Euler-Lagrange}$$

Variational Derivative — Gelfand, Fomin

$$S[y] = \int_a^b f(y, y'; x)\, dx$$

discretize $y(x)$



$x_{i+1} - x_i \equiv \Delta x$

$n+1$ intervals

$x_i, \; i = 0, n+1$

$y_i, \; i = 0, n+1$

then we have function and sum:

$$S(y_0, \ldots, y_{n+1}) = \sum_{n=0}^{n} f\left(y_i, \frac{y_{i+1} - y_i}{\Delta x}, x_i\right) \Delta x$$

n variables since y_0, y_{n+1} are fixed

$$\frac{\partial S}{\partial y_k} = \frac{\partial f}{\partial y}\left(y_k, \frac{y_{k+1} - y_k}{\Delta x}, x_k\right) \Delta x$$

$$+ \frac{\partial f}{\partial y'}\left(y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}, x_{k-1}\right)$$

$i = k-1$ term

$$- \frac{\partial f}{\partial y'}\left(y_k, \frac{y_{k+1} - y_k}{\Delta x}, x_k\right)$$

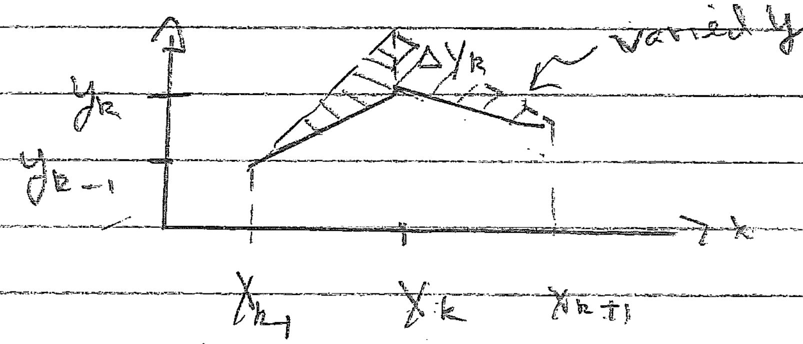
$i = k$ term

So

$$\frac{\partial S}{\partial y_k \Delta x} = \frac{\partial f}{\partial y}\left(y_k, \frac{y_{k+1}-y_k}{\Delta x}, x_k\right)$$

$$- \frac{1}{\Delta x}\left[\frac{\partial f}{\partial y'}\left(y_k, \frac{y_{k+1}-y_k}{\Delta x}, x_k\right) - \frac{\partial f}{\partial y'}\left(y_{k-1}, \frac{y_k - y_{k-1}}{\Delta x}, x_{k-1}\right)\right]$$

geometrically,



hatched area

$$\frac{A}{2} = \frac{\Delta x}{2}\left(y_k + \Delta y_k - y_{k-1}\right) - \frac{\Delta x}{2}\left(y_k - y_{k-1}\right)$$

$$= \frac{\Delta x \Delta y_k}{2}$$

Variational derivative:

$$\frac{\delta S[y]}{\delta y} \equiv \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \frac{\partial S}{\partial y_k \Delta x} = \underline{\underline{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}}}$$