Phys 330

## Lecture 12: Lagrangian

Lagrangian $\mathcal{L} \equiv T - U$

Actual path is the one which makes the action stationary

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt \;;\quad \delta S = 0 \quad \text{Hamilton's principle}$$

where path variations are consistent with all constraints.

Consider single point particle moving in 1D:

$$T = \tfrac{1}{2} m \dot{x}^2$$
$$\mathcal{L}(x, \dot{x}; t) = \tfrac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x}$$

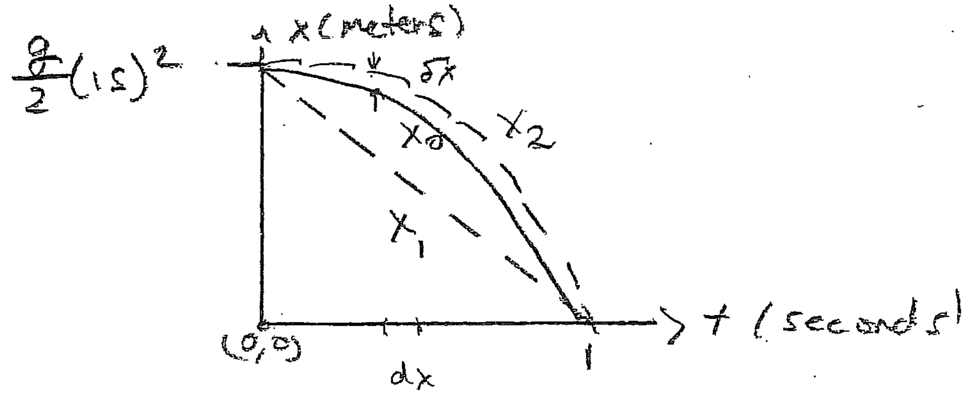
$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$F_x - m\ddot{x} = 0$$

In 3 Dimensions, get three Euler equations

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \qquad i = 1, 2, 3$$

A toy calculation - uniform acceleration




$\delta x$ is difference between paths at <u>same time</u>
in contrast to differential $dx = x(t+dt) - x(t)$

3 paths from $(x,t) = (\frac{g}{2}, 0) \to (0,1)$

$$X_0(t) = \tfrac{1}{2} g (1-t^2)$$
$$X_1(t) = \tfrac{1}{2} g (1-t)$$
$$X_2(t) = \tfrac{1}{2} g (1-t^3)$$

directly integrate to get $S$:

$$\dot{X}_0 = -gt \qquad U = mgx$$

$$\mathcal{L}_0 = \tfrac{1}{2} mg^2t^2 - \frac{mg^2}{2}(1-t^2) = \frac{mg^2}{2}(2t^2-1)$$

$$S_0 = \int_0^1 \mathcal{L}_0 \, dt = \frac{mg^2}{2}\left(\frac{2}{3}-1\right) = -\frac{mg^2}{2}\left(\frac{1}{3}\right) \quad \text{min}$$

Similarly, $S_1 = -\frac{mg^2}{2}\left(\frac{1}{4}\right)$
$S_2 = -\frac{mg^2}{2}\left(\frac{-3}{10}\right)$

$$\dim\left[mg^2(1s)^3\right] = kg\left(\frac{m}{s^2}\right)^2 s^3 = \frac{kg\, m^2}{s} = J \cdot s \quad [\text{energy} \times \text{time}]$$

same as Planck's constant

Generalized coordinates

System of $N_p$ point particles. Configuration of system is completely described by $3N_p$ coordinates. With $N_c$ equations of constraint, we have $N_f = 3N_p - N_c$ degrees of freedom. Thus we can specify the configuration of constrained system by $N_f$ generalized coordinates $q_i$. These coordinates are not necessarily cartesian. $\mathcal{L}$ then depends on $q_i$, generalized velocities $\dot{q}_i$, and possibly $t$:

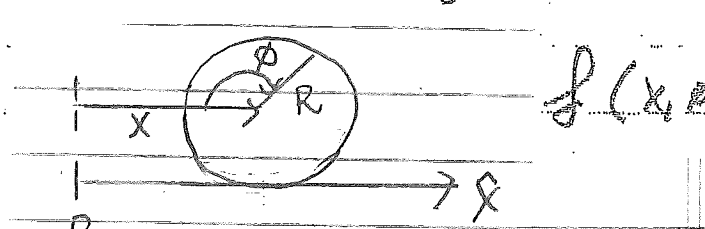
$$\mathcal{L}(q_i, \dot{q}_i; t)$$

Fine print: Constraint function that are relations between $q_i$ are called holonomic.

no velocities!

$$f_j(q_i; t) = 0 \quad j = 1, \ldots, N_c$$

Example: Disk rolling in 1 Dimension

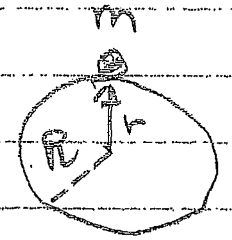



$$f(x, \phi) = x - R\phi = 0$$

rigid body coordinate $x$ of center of mass and one angle $\phi$.

## Nonholonomic constraints.

example 1: bead sliding on top of cylinder



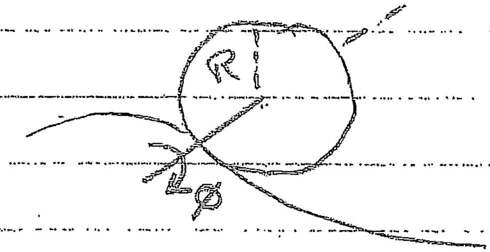

inequality:

$r^2 - R^2 \geq 0$

bead eventually slides off cylinder.

example 2: Rolling cylinder in two dimensions. (Goldstein)

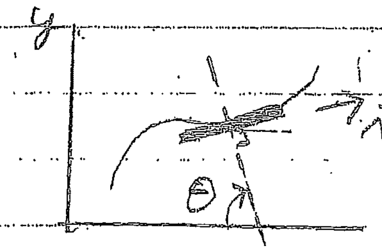
axis of cylinder




x, y path

Coordinates are: position of center of mass $(x,y)$, rotation about axis $\phi$, orientation of axis $(\theta)$

Looking down onto x-y plane




$$\vec{v} = v\sin\theta\,\hat{x} + v\cos\theta\,\hat{y}$$
$$= \dot{x}\,\hat{x} + \dot{y}\,\hat{y}$$

Constraints: $v = R\dot{\phi}$

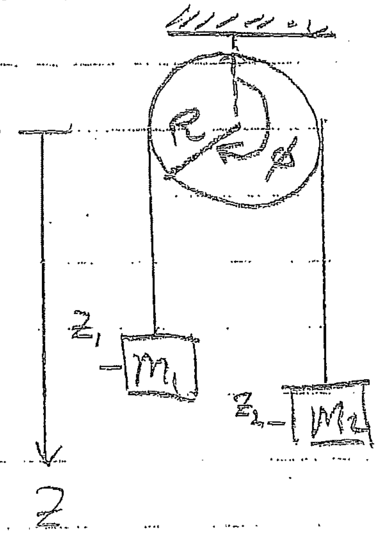
$\dot{x} - R\dot{\phi}\sin\theta = 0$ and $\dot{y} - R\dot{\phi}\cos\theta = 0$

Cannot be integrated $x(t), \phi(t), \theta(t)$

Note in 1D, $\dot{x} = r\dot{\phi}$
integrates to $x = r\phi + \text{const}$

Velocity dependent constraints are also nonholonomic. We will consider only holonomic constraints. For holonomic constraints, we can choose generalized coordinates equal to $N_f$, or introduce Lagrange multipliers to get forces of constraint.

Example: Atwood machine:




mass of disk $M_D$, moment of inertia $I_D = \frac{1}{2} M_D R^2$

Coordinates: $z_1, z_2, \phi$

Constraints: $z_1 + z_2 = l$ length of rope
$z_1 = \phi R$ does not slip.

$$T = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2 + \frac{1}{2} I_d \dot{\phi}^2$$

$$V = -g m_1 z_1 - g m_2 z_2$$

Use constraints to eliminate 2 coordinates:

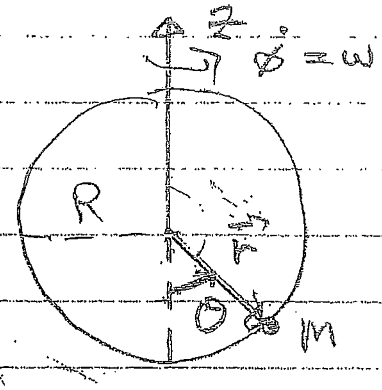
$$U = -g(m_1 - m_2) z_1 - g m_2 l$$

$$\dot{z}_2 = -\dot{z}_1 \; ; \; \dot{\phi} = \frac{\dot{z}_1}{R}$$

$$T = \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2} M_d\right) \dot{z}_1^2$$

gives $$\ddot{z}_1 = g\left[\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2} M_d}\right]$$

Example: rotating hoop. bead slides on wire hoop that is rotating with constant angular velocity $\omega$.




$U = mgR(1 - \cos\theta)$
with $U(0) = 0$

$$T \quad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$
$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2$$

constraints: $r - R = 0$
$\phi - \omega t = 0$

1 generalized coordinate $\theta$

$$v^2 = R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta$$

$$\mathcal{L} = \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta\right) - mgR(1-\cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgR\sin\theta + mR^2\omega^2\sin\theta\cos\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\ddot{\theta}$$

$$\ddot{\theta} = \sin\theta\left[\omega^2\cos\theta - \frac{g}{R}\right]$$

<u>equilibrium position</u> $\ddot{\theta} = 0$,

$$\cos\bar{\theta} = \frac{g}{R\omega^2}$$

Expand about equilibrium; $\theta = \bar{\theta} + \delta$

$$\cos(\bar{\theta}+\delta) \cong \cos\bar{\theta} - \sin\bar{\theta}\,\delta$$
$$\sin(\bar{\theta}+\delta) \cong \sin\bar{\theta} + \cos\bar{\theta}\,\delta$$

$$\ddot{\delta} = (s + c\delta)\left[\omega^2(c - s\delta) - \frac{g}{R}\right]$$

cancel

$$\ddot{\delta} = -\omega^2\sin^2\bar{\theta}\,\delta$$

$$= -\omega^2\left[1 - \frac{g^2}{R^2\omega^4}\right]\delta$$

Or, $\ddot{\theta} = \omega^2 \sin\theta \left[\cos\theta - \frac{g}{R\omega^2}\right] \equiv -\frac{\partial}{\partial\theta} U_{eff}$

$U_{eff} \equiv$ effective potential. then

$$\left.\frac{d^2 U_{eff}}{d\theta^2}\right|_{\bar{\theta}} = \omega^2 \frac{d}{d\theta}\left[\underbrace{\frac{g}{R\omega^2}}_{\equiv a} \sin\theta - \cos\theta \sin\theta\right]\Big|_{\bar{\theta}} > 0$$

$\cos\bar{\theta} = a$

$$a \cos\bar{\theta} + \sin^2\bar{\theta} - \cos^2\bar{\theta} > 0$$

$$a^2 + 1 - a^2 - a^2 > 0$$

$$\boxed{a < 1}$$

Stability:

if $\omega > \sqrt{\frac{g}{R}}$ we have a stable equilibrium.

if $\omega < \sqrt{\frac{g}{R}}$ then $g/R\omega^2 > 1$ and

$\therefore$ $\omega^2 \cos\theta - \frac{g}{R} = 0$ has no solution.

$$\ddot{\theta} = \sin\theta \left[\omega^2 \cos\theta - \frac{g}{R}\right]$$

we find stable solution for $\theta = 0$.

$$\ddot{\theta} \doteq -\theta \left[\frac{g}{R} - \omega^2\right]$$

There is another equilibrium point ($\theta = \pi$) which is unstable.

Stability plot (Newtonian Dynamics, Baierlein)