Lecture 13: Conservation Theorems; Lagrangian Examples

Conservation theorems (following Marion)

invariance $\Longleftrightarrow$ conservation law
Most elegantly expressed in (classical or quantum) field theory by Noether's theorem.

$\boxed{\text{① Energy}}$

repeated measurements give identical results in inertial frames. Inertial frames are homogeneous in time. time is absolute, same everywhere

Therefore for closed system $\frac{\partial \mathcal{L}}{\partial t} = 0$

no explicit time dependence.

$$\frac{d}{dt}\mathcal{L} = \sum_j \frac{\partial \mathcal{L}}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\ddot{q}_j$$

Euler eq. $= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)$

$$\frac{d}{dt}\mathcal{L} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\ddot{q}_j\right]$$

$$= \frac{d}{dt}\left[\sum_j \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\dot{q}_j\right]$$

giving

$$\frac{d}{dt}\left[\sum_j \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}\dot{q}_j - \mathcal{L}\right] = 0$$

$\equiv \mathcal{H}$ the Hamiltonian

$$\boxed{\text{with generalized momentum } p_j \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_j}}$$

$$\mathcal{H} = \sum_j \left(p_j \dot{q}_j - \mathcal{L} \right)$$

Legendre transformation

H(p,q;t)!

$$\boxed{\mathcal{H} \text{ is conserved if } \frac{\partial \mathcal{L}}{\partial t} = 0}$$

$\boxed{\text{Relation of } \mathcal{H} \text{ to } E.}$

Suppose $U(x_i)$ independent of $\dot{x}_i$ — no velocity dependence

and $x_i =$ x_i(q_1,...q_n) — transformation from generalized to Cartesian has no explicit time dependence

then $U(q_i)$ and $\frac{\partial U}{\partial \dot{q}_i} = 0$

generalized momentum $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}(T - U) = \frac{\partial T}{\partial \dot{q}_i}$

Now since $q_i = q_i$(x_1,...x_n) independent of t

$$T = \frac{1}{2}\sum_{i,j} A_{ij}\, \dot{q}_i \dot{q}_j$$

matrix $[A]$ not necessarily diagonal

$$P_k = \frac{\partial T}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i,j} A_{ij} \frac{\partial}{\partial \dot{q}_k}\left(\dot{q}_i \dot{q}_j\right)$$

$$= \frac{1}{2}\sum_i A_{ik}\dot{q}_i + \frac{1}{2}\sum_j A_{kj}\dot{q}_j$$

$$\sum_k P_k \dot{q}_k = \frac{1}{2}\sum_{ik} A_{ik}\dot{q}_i \dot{q}_k + \frac{1}{2}\sum_{ik} A_{ki}\dot{q}_i \dot{q}_k$$

$$= \sum_{i,k} A_{ik}\dot{q}_i \dot{q}_k = 2T$$

and therefore under these assuptions

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = 2T - T + U = T + U = E$$

<u>Note: Two separate questions</u>

① Is $\mathcal{H} = E$?

② Is $\mathcal{H}$ conserved? (constant in time)

Momentum Conservation

translation of system $\vec{r}_\alpha \to \vec{r}_\alpha + \vec{\varepsilon}$
where $\vec{\varepsilon}$ is small, fixed displacement

$$\vec{\varepsilon} = \sum \varepsilon_i \hat{e}_i$$

For simplicity, suppress particle index α,

$$\mathcal{L} = \mathcal{L}(x_i, \dot{x}_i)$$

$$\Delta \mathcal{L} = \sum \frac{\partial \mathcal{L}}{\partial x_i} \varepsilon_i + \sum \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \cancelto{0}{\dot{\varepsilon}_i} = 0$$

fixed displacement

since ε_i are arbitrary and independent.

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0$$

generalized momentum

$$\Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \Rightarrow p_i = \text{const}$$

Generalized momenta

p_i conserved if $\Delta \mathcal{L} = 0$ in displacement.
In fact, this is true for each p_i individually,

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \text{then } p_i = \text{constant}$$

generalized momentum = single particle momentum in simplest case

Check simple case:

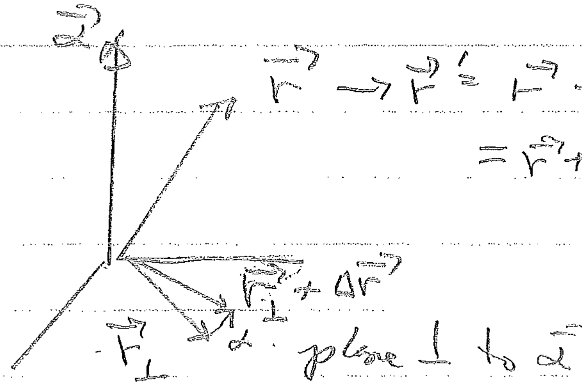
if $U(x_i)$ then

$$\frac{\partial}{\partial \dot{x}_i}(T-U) = \frac{\partial T}{\partial \dot{x}_i} \longrightarrow m\dot{x}_i$$

$$\tfrac{1}{2} m \dot{x}_i^2$$

<u>Conservation of Angular momentum:</u>

rotation by small constant angle $\vec{\alpha}$.



$$\Delta\vec{r} = \vec{\alpha} \times \vec{r}$$
$$\Delta\dot{\vec{r}} = \vec{\alpha} \times \dot{\vec{r}}$$

In Euclidean (rectangular coordinates)

$$\Delta\mathcal{L} = \sum \frac{\partial \mathcal{L}}{\partial x_i}\Delta x_i + \sum \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\Delta \dot{x}_i$$

$\dot{P}_i$ from Euler equation; P_i

$$\Delta\mathcal{L} = \sum\left[\dot{P}_i \Delta x_i + P_i \Delta\dot{x}_i\right]$$

$$\Delta \mathcal{L} = \frac{d}{dt}\sum\left(p_i \Delta x_i\right)$$

$$= \frac{d}{dt}\left[\vec{p}\cdot\left(\vec{\alpha}\times\vec{r}\right)\right]$$

since rotation is constant

$$= \vec{\alpha}\cdot\frac{d}{dt}\left[\vec{r}\times\vec{p}\right] = 0$$

$\vec{L} \equiv \vec{r}\times\vec{p}$ is conserved if

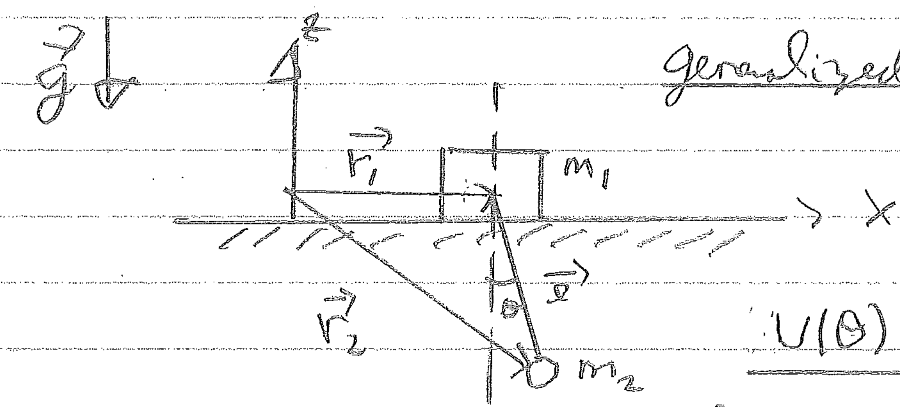
$\Delta \mathcal{L} = 0$ under rotation of coordinates.

$\boxed{\text{In summary:}}$

Invariance	conserved quantity
time	energy
translation	momentum
rotation	angular momentum

An example with a complicated kinetic energy.

Sliding pendulum



generalized coordinate: $x,\ \theta$

$$U(\theta) = m_2 g l(1-\cos\theta)$$

$$\vec{r}_2 = \vec{r}_1 + \vec{l} \qquad \vec{r}_1 = x\hat{x}\ ;\ \dot{\vec{r}}_1 = \dot{x}\hat{x}$$

$$= (x + \sin\theta\, l)\,\hat{x} - \cos\theta\, l\,\hat{y}$$

$$\dot{\vec{r}}_2 = (\dot{x} + \dot{\theta}\cos\theta\, l)\,\hat{x} + \dot{\theta}\sin\theta\, l\,\hat{y}$$

$$T = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left[(\dot{x}+\dot{\theta}\cos\theta\, l)^2 + \dot{\theta}^2\sin^2\theta\, l^2\right]$$

$$T(\dot{x},\dot{\theta},\theta) = \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_2\dot{\theta}\dot{x}\, l\cos\theta + \frac{1}{2}m_2 l^2\dot{\theta}^2$$

x equation of motion:

$$-\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\dot{\theta}\cos\theta = \text{Const.} = P_x$$

recall center of mass:

$$X_{cm} = \frac{m_1 x + m_2 (x + l \sin\theta)}{m_1 + m_2}$$

$$= x + \frac{m_2}{m_1+m_2} l \sin\theta$$

so the momentum of the center of mass is--

$$(m_1+m_2)\dot{X}_{cm} = (m_1+m_2)\dot{x} + m_2 l \dot{\theta} \cos\theta = P_x$$

Momentum of center of mass is constant.

θ equation of motion

$$\frac{\partial L}{\partial \theta} = \underbrace{-m_2 g l \sin\theta}_{U} \underbrace{- m_2 \dot{\theta} \dot{x} l \sin\theta}_{T}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 \dot{x} l \cos\theta + m_2 l^2 \dot{\theta}$$

before taking time derivative, keep in mind that

$$\dot{x} = \frac{1}{m_1+m_2} \left(P_x - m_2 l \dot{\theta} \cos\theta \right)$$

We will need this for the time derivative in the theta-equation.

But first...

We can simplify the algebra by carefull treatment of $\dot{x}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}(\dot{x})\, m_2 l \cos\theta + \dot{x}\, m_2 l(-\sin\theta)\dot{\theta}$$

$$+ \frac{d}{dt}(m_2 l^2 \dot{\theta})$$

then $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$ gives

$$\frac{d}{dt}(\dot{x})\, m_2 l \cos\theta + \frac{d}{dt}(m_2 l^2 \dot{\theta}) + \dot{\theta}\dot{x}\, m_2 l(-\sin\theta)$$

$$= -m_2 g l \sin\theta - m_2 \dot{\theta}\dot{x} l \sin\theta$$

✓ terms cancel + $\frac{d}{dt}(\dot{x}) = \frac{1}{m_1+m_2}\frac{d}{dt}(p_x - m_2 l \dot{\theta}\cos\theta)$

and using x-equation to explicitly calculate x double-dot

$$= \frac{-m_2 l}{m_1+m_2}\frac{d}{dt}(\dot{\theta}\cos\theta)$$

we get

$$-\frac{m_2^2 l^2}{m_1+m_2}\cos\theta \frac{d}{dt}(\dot{\theta}\cos\theta) + m_2 l^2 \frac{d}{dt}(\dot{\theta})$$

$$= -m_2 g l \sin\theta$$

final result for theta-equation of motion--

$$-\frac{m_2}{m_1+m_2}\cos\theta \frac{d}{dt}(\dot{\theta}\cos\theta) + \frac{d}{dt}\dot{\theta} = -\frac{g}{l}\sin\theta$$

for small θ, $\cos\theta \approx 1$, $\sin\theta \approx \theta$

$$\ddot{\theta}\left(1 - \frac{m_2}{m_1+m_2}\right) = -\frac{g}{\ell}\sin\theta$$

$$\omega = \sqrt{\frac{g(m_1+m_2)}{\ell\, m_1}}$$

<u>note</u>: if $m_1 \to \infty$ we find from original equation

$$\frac{d}{dt}\dot{\theta} = -\frac{g}{\ell}\sin\theta$$

recover simple pendulum.