Lecture 14: Method of Lagrange undetermined multipliers

From Euler equation with constraint we found

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \lambda(t)\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} + \lambda(t)\frac{\partial f}{\partial y} = 0$$

where the constraint $f(x,y,t)=0$.
but does not depend on velocities (is holonomic).

To see the meaning of λ, take the simple case

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - U(x,y)$$

$$-\frac{\partial U}{\partial x} + \lambda\frac{\partial f}{\partial x} = m_1\ddot{x}$$

$$-\frac{\partial U}{\partial y} + \lambda\frac{\partial f}{\partial y} = m_2\ddot{y}$$

$\lambda \frac{\partial f}{\partial x}$ is force of constraint on mass 1
$\lambda \frac{\partial f}{\partial y}$ is force of " " mass 2

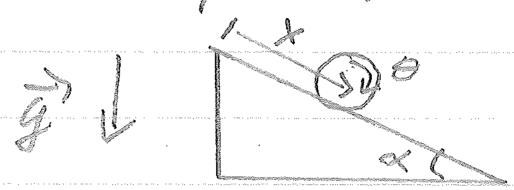
More generally, we have generalized coordinates q_i and f_k equations of constraint

$$Q_i = \sum_k \lambda_k \frac{\partial f_k}{\partial q_i}$$

are the generalized forces of constraint.

Example 1: Disk rolling down inclined plane (mass m, radius a)

$I = \frac{1}{2} m a^2$



l length of hypotenuse

$U = mg(l-x)\sin\alpha$ zero chosen at bottom

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} m \left(\dot{x}^2 + \frac{1}{2} a^2 \dot{\theta}^2\right)$$

constraint rolls w/o slipping: $x - a\theta = 0$

To find acceleration, use constraint to eliminate θ:

$$\mathcal{L} = \frac{1}{2} m \left(1 + \frac{1}{2}\right) \dot{x}^2 - mg(l-x)\sin\alpha$$

$$\frac{3}{2} m \ddot{x} = + mg \sin\alpha$$

$$\ddot{x} = + \frac{2}{3} g \sin\alpha$$

To find forces of constraint:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i}$$

$$m\ddot{x} = +mg\sin\alpha + \lambda \underbrace{\frac{\partial f}{\partial x}}_{=1} \qquad (1)$$

$$\frac{1}{2}ma^2\ddot{\theta} = \lambda \underbrace{\frac{\partial f}{\partial \theta}}_{=-a} \qquad (2)$$

and $x = a\theta$ $\qquad (3)$

3 eq. for unknowns $x(t), \theta(t), \lambda(t)$.

(3)+(2) gives $\frac{1}{2}m\ddot{x} = -\lambda$

+(1) gives $\frac{3}{2}\cancel{m}\ddot{x} = +\cancel{m}g\sin\alpha$

$$\ddot{x} = +\frac{2}{3}g\sin\theta$$

solve for lambda

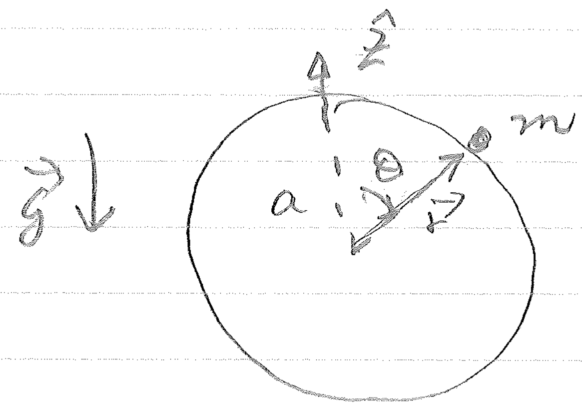
$$Q_x = \lambda\frac{\partial f}{\partial x} = -\frac{1}{3}mg\sin\alpha \quad \text{\underline{force}}$$

$$Q_\theta = \lambda\frac{\partial f}{\partial \theta} = -\frac{1}{3}mg\sin\alpha(-a)$$

$$= \frac{1}{3}mag\sin\alpha \quad \text{\underline{torque}}$$

Example 2 (Marion)

Bead sliding off disk:



the constraint $r - a \geq 0$ is nonholonomic.
But we can consider the holonomic constraint $r - a = 0$ and calculate when $\lambda = 0$.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$U = mgr\cos\theta$$

$$m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda \overset{1}{\cancel{\frac{\partial f}{\partial r}}}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = mgr\sin\theta + \lambda \overset{0}{\cancel{\frac{\partial f}{\partial \theta}}}$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = gr\sin\theta$$

plus constraint equation

$$r = a \; ; (\dot{r} = 0)$$

r eq.) $\dot{\theta}^2 = \frac{g}{a}\cos\theta - \frac{\lambda}{ma}$

θ eq.) $\ddot{\theta} = \frac{g}{a}\sin\theta$

$$\frac{d}{dt}(\dot{\theta}^2) = 2\,\dot{\theta}\,\ddot{\theta}$$

and acting on any function $f(\theta(t))$

$$\frac{df}{dt} = \frac{d\theta}{dt}\,\frac{df}{d\theta} = \dot{\theta}\,\frac{df}{d\theta}$$

$$\dot{\theta}\,\frac{d}{d\theta}(\dot{\theta}^2) = 2\dot{\theta}\ddot{\theta} \underset{\theta\text{ eq.}}{=} 2\dot{\theta}\left[\frac{g}{a}\sin\theta\right]$$

so $\frac{d}{d\theta}(\dot{\theta}^2) = \frac{2g}{a}\sin\theta$ integrate

$$\dot{\theta}^2(\theta_0) = \int_0^{\theta_0} \frac{2g}{a}\sin\theta\,d\theta = \frac{2g}{a}(1-\cos\theta_0)$$

$$\underset{r\text{ eq.}}{=} \frac{g}{a}\cos\theta - \frac{\lambda(\theta_0)}{ma}$$

solve for $\lambda(\theta_0)$:

$$\lambda(\theta_0) = mg\,(3\sin\theta_0 - 2)$$

Value of θ_0 giving $\lambda = 0$ is

$$\cos\theta_0 = 2/3$$

$$\theta_0 = 48.2^\circ$$