Physics 303, Fall 2013

updated Oct. 18, 2017

Lecture 15: Hamiltonian Dynamics

Recall generalized momentum,

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

So Lagrange equation is:

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

We found that if $\frac{\partial \mathcal{L}}{\partial t} = 0$ the Hamiltonian

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

was conserved. For $U(x_i)$ and $q_i = q_i(x_i)$

then $\frac{\partial U}{\partial \dot{q}_i} = 0$ & $T = \frac{1}{2}\sum_{i,j} A_{ij}\dot{q}_i\dot{q}_j$

And then $\mathcal{H} = T + U = E$

Mathematically, the transformation from $\mathcal{L} \to \mathcal{H}$ is a Legendre Transformation

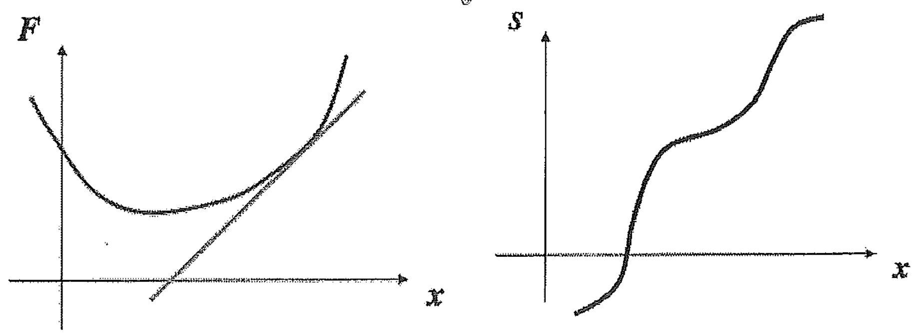
$$\mathcal{L}(q_i, \dot{q}_i, t) \longleftrightarrow \mathcal{H}(q_i, p_i, t)$$

a special kind of change of variables

What is the big deal, since in the simplest case p=m*x-dot (multiplication by a constant)?

$dG = -F'dx + s\,dx + x\,ds$

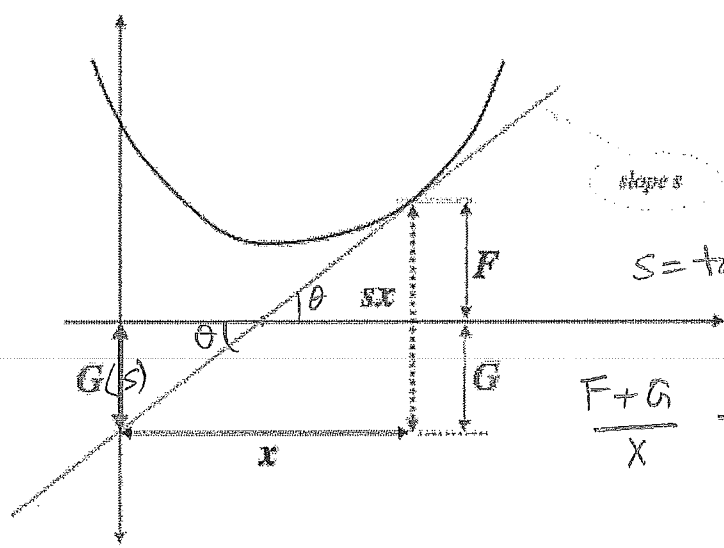
From: "Making Sense of the Legendre Transform", Zia, Redish, McKay ,*arXiv:0806.1147v2 4 March 2009.* and also Gelfand & Fomin p.72



Convex function $F(x)$ and slope

$$s(x) = \frac{dF}{dx}$$

. F convex means $s(x)$ can be inverted to get $x(s)$.



$s = \tan\theta$

$\frac{F+G}{x} = \tan\theta = s$

Graphical representation of the relation

$$xs = F + G$$

. The function $G(s)$ contains the same information as $F(x)$. Furthermore,

$$x(s) = \frac{dG}{ds}$$

because $d(xs) = s(x)\,dx + x(s)\,ds = \frac{dF}{dx}dx + \frac{dG}{ds}ds$

$G(s) = xs - F$ & $F(x) = xs - G$

Consider $f(x,y)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \equiv u\,dx + w\,dy$$

where in general $u(x,y)$, $w(x,y)$ are functions of x, y
Change variables from (x,y) to (u,y).

Legendre transform is

$$g = ux - f$$

then
$$dg = x\,du + u\,dx - \underbrace{\frac{\partial f}{\partial x}}_{=u}dx - \underbrace{\frac{\partial f}{\partial y}}_{=w}dy$$

cancel

$$dg = x\,du - w\,dy \equiv \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy$$

so $x = \dfrac{\partial g}{\partial u}$ and $g(u,y)$

we construct g by <u>inverting</u> $u(x)$
to get $x(u)$ and then

$$g(u,y) = u\,x(u) - f(x(u), y)$$

trivial case : $\dot{x} = p/m$

$$\mathcal{H} = p\left(\frac{p}{m}\right) - \left[\frac{1}{2}m\left(\frac{p}{m}\right)^2 - V(x)\right] = \frac{1}{2}\frac{p^2}{m} + V(x)$$

Legendre transform is an involution gives identity applied to itself

Example: Special Relativity

set $c=1$. $H = \sqrt{p^2+m^2}$

transform to $\mathcal{L}(x,v)$: where v=x-dot.
so we swap p for x-dot:

$$\mathcal{L} = pv - H$$

we will see that the velocity is

$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$

in this case $v = \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{p^2+m^2}}$

inverting, $v^2 = \frac{p^2}{p^2+m^2}$

The form of p(v) is completely determined by H! $p = \frac{vm}{\sqrt{1-v^2}}$

$$\mathcal{L} = \frac{v^2 m}{\sqrt{1-v^2}} - \left[\frac{v^2m^2}{1-v^2} + \frac{m^2(1-v^2)}{1-v^2}\right]^{1/2}$$

$$= \frac{v^2m}{\sqrt{1-v^2}} - \frac{m}{\sqrt{1-v^2}} = -\frac{m(1-v^2)}{\sqrt{1-v^2}}$$

$$\boxed{\mathcal{L} = -m\sqrt{1-v^2}} \underset{v\ll 1}{\simeq} -m\left(1-\tfrac{1}{2}v^2\right)$$

$$= -m + \tfrac{1}{2}mv^2$$

$\boxed{\text{Hamilton's Equations}}$

$$H = \sum p_i \dot{q}_i - \mathcal{L}$$

$$dH = \sum_i \left(p_i \, d\dot{q}_i + \dot{q}_i \, dp_i\right)$$

$$- \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i \right)$$

$$dH = \sum_i \left[\dot{q}_i \, dp_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i \right]$$

also, $dH = \sum \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$ because H(p,q)

We therefore have:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \text{Lagrange equation}$$

$$\frac{\partial H}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\dot{p}_i$$

$$\boxed{\begin{aligned} &(1) \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \\ &(2) \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i \end{aligned}}$$

Hamilton's equation: each 2nd order diff eq. replaced by two 1st order diff eq.

Notice minus sign goes with derivative of potential equation.
If (2) is "F=dP/dt", why do we need (1)?

Derivation of Hamilton's equations from Hamilton's principle.

Hamilton's Principle

$$S = \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H \right) dt$$

$$\delta S = \int_{t_1}^{t_2} \sum_i \left\{ p_i \delta\dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i \right\} dt$$

integrate by parts $\int p_i \frac{d}{dt}\delta q_i = -\int \dot{p}_i \delta q_i$

collect terms.

$$\delta S = 0 = \int \sum_i \left\{ -\left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i + \left(-\dot{q}_i + \frac{\partial H}{\partial p_i}\right)\delta p_i \right\} dt$$

Here δq_i, δp_i are independent variations.

Hamilton's Equations follow. Here, relations between q_i, p_i are the equations of motion in contrast to $\mathcal{L}$ approach where $\dot{q}_i = \frac{d}{dt} q_i$

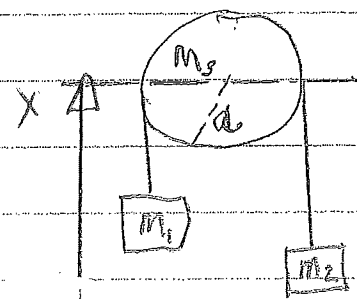
Time Dependence of H.

$$\frac{dH}{dt} = \underbrace{\sum_i \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \right)}_{0 \text{ from equations of motion}} + \frac{\partial H}{\partial t}$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

If $\frac{\partial H}{\partial t} = 0$ and $q_i = q_i(x_i)$ do not depend on time then $H = T + U$. These q_i are called "natural".

Example Atwood machine



$I = \frac{1}{2} m a^2$

$\dot{x} = a\dot{\theta}$

$x_1 + x_2 = \text{constant}$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}\left(\frac{1}{2}m_3 a^2\right)\dot{\theta}^2$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}m_3\right)\dot{x}_1^2$$

$$U = +m_1 g x_1 + m_2 g x_2 = +(m_1 - m_2) g x_1 + \text{const}$$

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \left(m_1 + m_2 + \frac{m_3}{2}\right)\dot{x}_1 \equiv m_{eff}\dot{x}_1$$

$$H = P_x \dot{x}_1 - T + U = \frac{P_x^2}{m_{eff}} - \frac{1}{2}\frac{P_x^2}{m_{eff}} + U$$

$$= \frac{1}{2}\frac{P_x^2}{m_{eff}} + U = T + U$$

as expected

You can skip this if q_i are natural and write $H = T + U$ directly

$$\dot{x}_1 = \frac{\partial H}{\partial p_x} = \frac{p_x}{m_{eff}}$$ relates velocity to momentum

$$\dot{p}_x = -\frac{\partial H}{\partial x_1} = -(m_1 - m_2) g$$

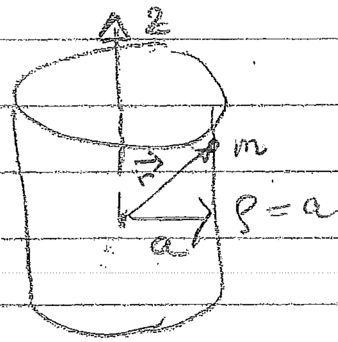
combine to get

$$\ddot{x}_1 = \frac{\dot{p}_x}{m_{eff}} = -\left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2} m_3}\right) g$$

Note: for complicated problems, the first order equations are easily integrated numerically

$\boxed{\text{Example: Particle on cylinder (13.13)}}$

constraint
$r^2 = a^2 + z^2$



Cylindrical coordinates ρ, ϕ, z

$\vec{F} = -k r \hat{r}$

$$v^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 = a^2 \dot{\phi}^2 + \dot{z}^2$$

$$T = \frac{1}{2} m \left(a^2 \dot{\phi}^2 + \dot{z}^2\right)$$

$$U = +\frac{k}{2} r^2 = +\frac{k}{2}\left(a^2 + z^2\right) = +\frac{k}{2} z^2 + \text{const}$$

Applying the constraint, we have 2 degrees of freedom, here taken as z and phi.

Generalized momenta are: and related to velocities as

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = ma^2\dot\theta \quad (= I\omega)$$

$$P_z = \frac{\partial \mathcal{L}}{\partial \dot z} = m\dot z \qquad (*)$$

$$H = T + U = \frac{P_\phi^2}{2ma^2} + \frac{P_z^2}{2m} + \frac{1}{2}kz^2$$

$$\dot P_\theta = -\frac{\partial H}{\partial \theta} = 0$$ ignorable or cyclic coordinate

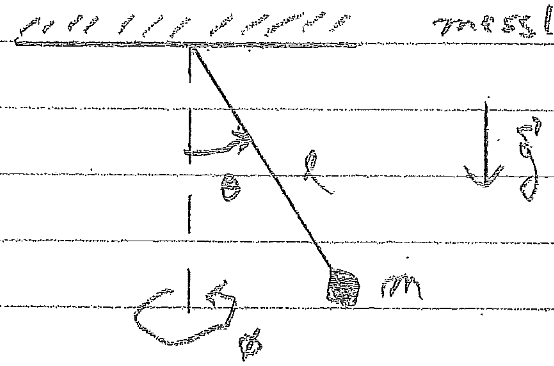
$$\boxed{P_\theta = \text{constant} = ma^2\dot\theta}$$ angular momentum is conserved

$$\dot P_z = -kz = \frac{d}{dt}(m\dot z)$$ last equality uses (*)

$$\boxed{\ddot z = -\frac{k}{m}z}$$

Example:

$\boxed{\text{Spherical Pendulum:}}$ Mass suspended by rigid, massless rod.



$$U = -mgl\cos\theta$$ -mgl to +mgl

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$
$$= l^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$r = l =$ constant

$$T = \frac{1}{2} m l^2 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2)$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = ml^2\dot\theta$$ generalize momenta

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = ml^2\sin^2\theta\,\dot\phi$$

notice $\mathcal{L} = T - U$ invariant under $\phi \to \phi + \phi_0$

here we have $H = T + U$

$$= \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta$$

eq. of motion:

(1) $$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$ momenta related to velocities

(1') $$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ml^2\sin^2\theta}$$

(2) $$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{m l^2 \sin^3\theta} - mgl\sin\theta$$

(2') $$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \underline{\text{cyclic}}$$

$$p_\phi = m l^2 \sin^2\theta\,\dot{\phi} = \text{Const} \quad \text{conserved}$$

$$\dot{\phi} = \frac{p_\phi}{m l^2}\,\frac{1}{\sin^2\theta}$$

(1) & (2) $$\ddot{\theta} = \frac{\dot{p}_\theta}{m l^2} = \left(\frac{p_\phi^2}{m^2 l^4}\right)\frac{\cos\theta}{\sin^3\theta} - \frac{g}{l}\sin\theta$$

We also have energy conservation,

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2}\frac{p_\phi^2}{m l^2}\frac{1}{\sin^2\theta} - mgl\cos\theta$$

$$\frac{1}{2}\dot{\theta}^2 + \frac{b^2}{2\sin^2\theta} - \frac{g}{\ell}\cos\theta = \frac{E}{m\ell^2} \equiv E'$$

$$b = \frac{p_\phi}{m\ell^2} = \dot{\phi}\Big|_{\theta=\pi/2}$$

let $c = \cos\theta$ $\quad \dot{c} = -\sin\theta\,\dot{\theta}$

$$\frac{1}{\sin^2\theta} = \left(\frac{\dot{\theta}}{\dot{c}}\right)^2$$

$$\frac{1}{2}\dot{\theta}^2 + \frac{b^2}{2}\frac{\dot{\theta}^2}{\dot{c}^2} - \frac{g}{\ell}c = E'$$

$$\frac{1}{\dot{\theta}^2} = \frac{\sin^2\theta}{\dot{c}^2} = \frac{(1-c^2)}{\dot{c}^2}$$ divide by $\dot{\theta}^2$

$$\frac{1}{2} + \frac{b^2}{2}\frac{1}{\dot{c}^2} - \left[\left(\frac{g}{\ell}\right)c + E'\right]\frac{1-c^2}{\dot{c}^2} = 0$$

$$\frac{1}{2}\dot{c}^2 + \frac{b^2}{2} - (1-c^2)\left[\frac{g}{\ell}c + E'\right] = 0$$

$$\dot{c}^2 = 2(1-c^2)\left[\frac{g}{\ell}c + E'\right] - b^2 \equiv f(c)$$

integrate to get

$$t = \int_{c_1}^{\cos\theta} \frac{dc}{\sqrt{f(c)}}$$

Spherical pendulum

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} \frac{P_\phi^2}{m l^2} \frac{1}{\sin^2\theta} - mgl\cos\theta$$

$$\frac{1}{2}\dot{\theta}^2 + \frac{b^2}{2\sin^2\theta} - \frac{g}{l}\cos\theta = \frac{E}{ml^2} \equiv E'$$

$$b = \frac{P_\phi}{ml^2} = \dot{\phi}\Big|_{\theta=\pi/2}$$

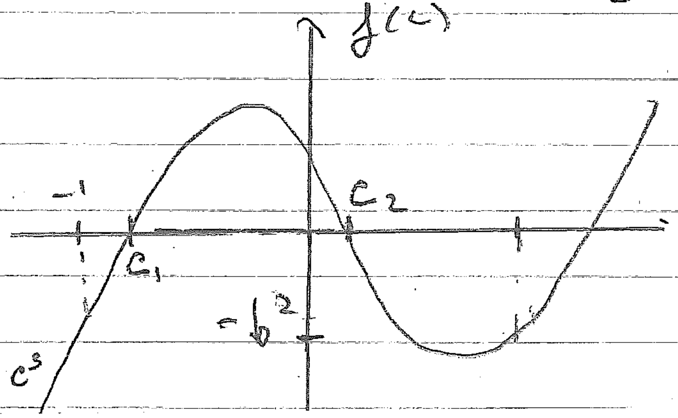
let $c = \cos\theta$, get

$$\dot{c}^2 = 2(1-c^2)\left[\frac{g}{l}c + E'\right] - b^2 \equiv f(c)$$

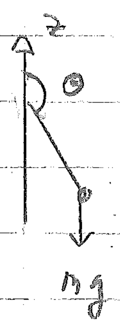
$$t(\cos\theta) = \int_{c_1}^{\cos\theta} \frac{dc}{\sqrt{f(c)}}$$

f(c) always positive for some region in [-1,1].

J. Lowenstein
Essentials of Hamiltonian Dynamics
Cambridge, 2012



here, θ defined



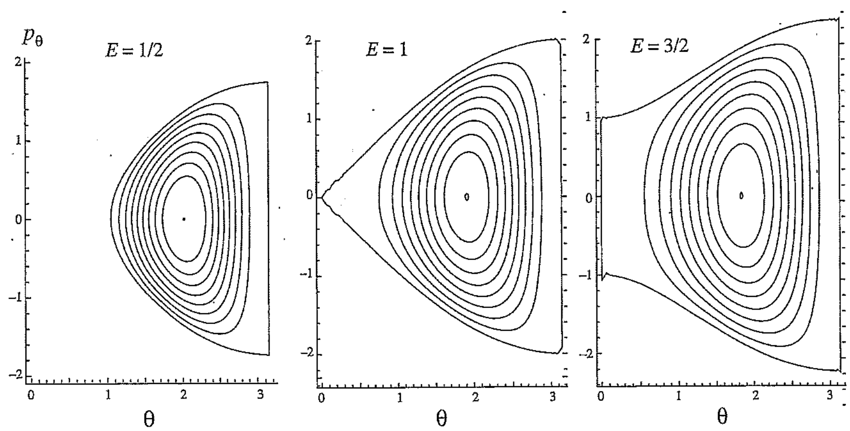


Figure 3.16 Projected orbits in the θ, p_θ plane for $L \geq 0$, $E = \frac{1}{2}, 1, \frac{3}{2}$. Each closed contour corresponds to a distinct value of L. The projected orbits for $L \leq 0$ look the same. from Lowenstein, Essentials of Hamiltonian Dynamics 2012

Canonical Transformation

notation $\bar{q} \equiv (q_1, \dots q_N)$ $\bar{p} \equiv (p_1, \dots p_N)$

transformation $\bar{Q} = (\bar{q}, \bar{p})$; $\bar{P} = (\bar{q}, \bar{p})$ is canonical if there is an $H'(\bar{Q}, \bar{P})$ such that

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} \quad ; \quad \dot{P}_i = -\frac{\partial H'}{\partial Q_i}$$

example 13.24 let $Q = p$, $P = -q$

then $H' = H$

and

$$\frac{\partial H}{\partial P} = \dot{Q} \Rightarrow \frac{\partial H}{\partial q} = -\dot{p}$$

$$\frac{\partial H}{\partial Q} = -\dot{P} \Rightarrow \frac{\partial H}{\partial p} = \dot{q}$$

H.W. $q = \sqrt{2P} \sin Q$; $p = \sqrt{2P} \cos Q$

Charged Particle in Magnetic Field

$V(\vec{r})$ scalar potential
$\vec{A}(\vec{r})$ vector potential

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Gauge invariance. $\Lambda(\vec{r},t)$ arbitrary

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \qquad V \to V' = V - \frac{\partial \Lambda}{\partial t}$$

since $\vec{\nabla}\times\vec{\nabla}\Lambda = 0$ $\quad \vec{B}' = \vec{B}$ and

$$\vec{E}' = -\vec{\nabla}V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}V + \frac{\partial}{\partial t}\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}V = \vec{E}$$

Lagrangian: $\mathcal{L} = \frac{1}{2}m|\dot{\vec{r}}|^2 - q(V - \dot{\vec{r}}\cdot\vec{A})$ see next page

Under gauge transformation

$$S \to S' = \int -q(V' - \dot{\vec{r}}\cdot\vec{A}')\,dt$$

$$= -q\int \left(V - \frac{\partial \Lambda}{\partial t} - \dot{\vec{r}}\cdot\vec{A} - \dot{\vec{r}}\cdot\vec{\nabla}\Lambda\right) dt$$

$$= S + q\int_{t_1}^{t_2} \underbrace{\left(\frac{\partial}{\partial t} + \dot{\vec{r}}\cdot\vec{\nabla}\right)}_{=\frac{d}{dt}\text{ total derivative along trajectory}}\Lambda\, dt$$

$$= S + q\left[\Lambda(\vec{r}_2, t_2) - \Lambda(\vec{r}_1, t_1)\right] = S$$

with Λ = constant at endpoints

$$\mathcal{L} \xrightarrow[v\to 0]{} -q\bar{V}$$

4-vector potential $A^\mu = (V, \vec{A})$

4-velocity $U^\mu = c\gamma(1, \vec{v}/c)$

Lorentz invariant, translationally invariant

$$\frac{1}{c\gamma} U^\mu A_\mu = V - \frac{\vec{v}}{c}\cdot\vec{A}$$

So

$$\mathcal{L} = -mc^2\sqrt{1-(v/c)^2} - q\bar{V} + \frac{q}{c}\vec{v}\cdot\vec{A}$$

$$\cong -mc^2 + \frac{1}{2}mv^2 - q\bar{V} + \frac{q}{c}\vec{v}\cdot\vec{A}$$

<u>Lorentz force law</u>

$$\frac{\partial \mathcal{L}}{\partial r_i} = -q\frac{\partial V}{\partial r_i} + q\,\dot{\vec{r}}\cdot\left(\frac{\partial \vec{A}}{\partial r_i}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = m\dot{r}_i + qA_i$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = m\ddot{r}_i + q\sum_j \frac{\partial A_i}{\partial r_j}\dot{r}_j + q\frac{\partial A_i}{\partial t}$$

$$= \frac{\partial \mathcal{L}}{\partial r_i} = -q\frac{\partial V}{\partial r_i} + q\,\dot{\vec{r}}\cdot\left(\frac{\partial \vec{A}}{\partial r_i}\right)$$

look at $i=1$ equation,

$$m\ddot{r}_1 = \underbrace{-q\frac{\partial V}{\partial r_1} - q\frac{\partial A_1}{\partial t}}_{qE_1} + q\left[\dot{r}_1\frac{\partial A_1}{\partial r_1} + \dot{r}_2\frac{\partial A_2}{\partial r_1} + \dot{r}_3\frac{\partial A_3}{\partial r_1}\right]$$

$$-q\left[\frac{\partial A_1}{\partial r_1}\dot{r}_1 + \frac{\partial A_1}{\partial r_2}\dot{r}_2 + \frac{\partial A_1}{\partial r_3}\dot{r}_3\right.$$

$$= qE_1 + q\dot{r}_2\underbrace{\left(\frac{\partial A_2}{\partial r_1} - \frac{\partial A_1}{\partial r_2}\right)}_{B_3} - q\dot{r}_3\underbrace{\left(\frac{\partial A_1}{\partial r_3} - \frac{\partial A_3}{\partial r_1}\right)}_{B_2}$$

$$= qE_1 + q\left(\dot{\vec{r}}\times\vec{B}\right)_1$$

$$m\ddot{\vec{r}} = q\left(\vec{E} + \dot{\vec{r}}\times\vec{B}\right)$$

Hamiltonian

canonical momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{r}_i} = m\dot{r}_i + qA_i$$

invert, $$\dot{\vec{r}} = \frac{\vec{p} - q\vec{A}}{m}$$ $\longrightarrow (\vec{p} - q\vec{A} + q\vec{A})$

$$H = \vec{p}\cdot\dot{\vec{r}} - \mathcal{L} = \vec{p}\cdot\left(\frac{\vec{p}-q\vec{A}}{m}\right)$$

$$-\frac{m}{2}\left|\frac{\vec{p}-q\vec{A}}{m}\right|^2 + q\left[V - \left(\frac{\vec{p}-q\vec{A}}{m}\right)\cdot\vec{A}\right]$$

$$= \frac{1}{2m}\left|\vec{p}-q\vec{A}\right|^2 + q\vec{A}\cdot\left(\frac{\vec{p}-q\vec{A}}{m}\right) + qV - q\vec{A}\cdot\left(\frac{\vec{p}-q\vec{A}}{m}\right)$$

$$H = \frac{1}{2m}\left|\vec{p}-q\vec{A}\right|^2 + qV$$

Check Hamilton's equations -

$$\dot{r}_i = \frac{\partial H}{\partial p_i} = \frac{p_i - qA_i}{m}$$

$$-\dot{p}_i = \frac{\partial H}{\partial r_i} = \frac{q}{m}\underbrace{(\vec{p}-q\vec{A})}_{m\dot{\vec{r}}}\cdot\frac{\partial\vec{A}}{\partial r_i} + q\frac{\partial V}{\partial r_i}$$

$$m\ddot{r}_i = \dot{p}_i - q\dot{A}_i = -q\frac{\partial V}{\partial r_i} + q\,\dot{\vec{r}}\cdot\left(\frac{\partial\vec{A}}{\partial r_i}\right)$$

$$-q\sum_j \frac{\partial A_i}{\partial r_j}\dot{r}_j - q\frac{\partial A_i}{\partial t}$$

which we recognize from before, $m\ddot{r}_i = qE_i + q(\dot{\vec{r}}\times\vec{B})_i$

the Lorentz force law

Two more problems from chapter 13

13.22

$$H = p\dot{q}(p,q) - \mathcal{L}(q, \dot{q}(p,q))$$

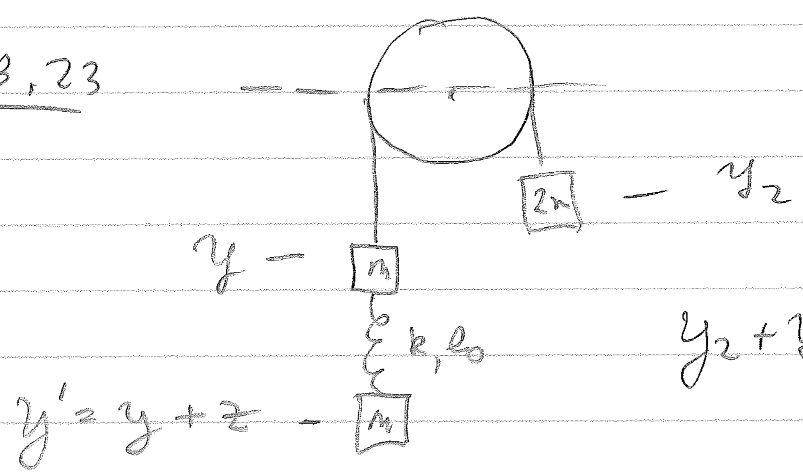
$$\frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}$$

$$= \left(p - \frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q}$$

cyclic same as ignorable

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow \dot{p} = 0$$

13.23



$y_2 + y = \text{const}$

$$U = mgy + mgy' + 2mg\underbrace{y_2}_{\text{const} - y} + \tfrac{1}{2}k(y' - y - l_0)^2$$

$$= mg(y' + y - 2y) + \tfrac{1}{2}k(y' - y - l_0)^2 + \text{const}$$

$$\boxed{U(z) = mgz + \tfrac{1}{2}k(z - l_0)^2}$$

$$\frac{dU}{dz} = -mg + k(z - l_0) \; ; \; \frac{d^2U}{dz^2} = k \; ; \; \frac{d^3U}{dz^3} = 0$$

$$U(z_{eq} + x) = U(z_{eq}) + \frac{1}{2}U''(z_{eq})\, x^2$$

$$= U(z_{eq}) + \frac{1}{2}kx^2 \qquad \text{ignore constant}$$

$$y' = y + z_{eq} + x \qquad y_1 + y_2 = 0$$

$$\dot{y}' = \dot{y} + \dot{x} \qquad \dot{y} = -\dot{y}_2$$

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{y} + \dot{x})^2 + \frac{1}{2}(2m)\dot{y}^2$$

$$P_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m(4\dot{y} + \dot{x}) \; ; \; P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(\dot{y} + \dot{x})$$

$$P_y - P_x = 3m\dot{y}$$

$$H = \frac{1}{2m}\left(\frac{1}{3}\right)(P_y - P_x)^2 + \frac{1}{2m}P_x^2 + \frac{1}{2}kx^2$$

P_y is cyclic: $\dot{P}_y = -\frac{\partial H}{\partial y} = 0$; $\dot{y} = \frac{\partial H}{\partial P_y} = \frac{1}{3m}(P_y - P_x)$

already know

$$\dot{P}_x = -\frac{\partial H}{\partial x} = -kx$$

$$\dot{x} = \frac{\partial H}{\partial P_y} = -\frac{1}{3m}(P_y - P_x) + \frac{P_x}{m}$$

$$\boxed{\ddot{x} = \frac{4}{3m}\dot{P}_x = -\frac{4k}{3m}x}$$

$$\omega = \sqrt{\frac{4k}{3m}}$$

Solution (c)

$$\underline{\underline{X(t)}} = X_0 \cos\omega t + \frac{V_{x0}}{\omega} \sin\omega t$$

$$\frac{P_y}{m} = 4\dot{y} + \dot{x}$$

$$\dot{y} = \frac{1}{4}\frac{P_y}{m} - \frac{\dot{x}}{4} \quad \text{and} \quad V_{y0} = \dot{y}(0) = \frac{1}{4}\frac{P_y}{m} - \frac{V_{x0}}{4}$$

integrating,

$$y(t) = \frac{1}{4}\frac{P_y t}{m} - \frac{1}{4}(x - X_0) + y_0$$

$$\underline{\underline{y(t)}} = \left(V_{y0} - \frac{V_{x0}}{4}\right)t - \frac{1}{4}(x - x_0) + y_0$$

$\underline{\underline{\text{take}}}$ $V_{x0} = 0$, $V_{y0} = 0$ so, $P_y = 0$, $\dot{y} = -\frac{1}{4}\dot{x}$

$$X(t) = X_0 \cos\omega t$$

$$y(t) = -\frac{1}{4}(X(t) - X_0) + y_0$$