Lecture 16: Phase Space

Point in phase space ξ "X_i" (our book uses z, but ξ is more traditional and fun to write).
over-bar means n-tuple, array of numbers.

$$\bar{\xi} = (\bar{q}, \bar{p}) = (q_1 \ldots q_N ; p_1 \ldots p_N)$$

N is number of degrees of freedom of system.
Initial conditions of system define a unique point in phase space. Time evolution gives phase space orbit; phase space orbits never cross if H is conserved.

Simple Example 1

Harmonic oscillator: $T = \frac{1}{2} m \dot{x}^2$

$$U = \frac{1}{2} m \omega^2 x^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\mathcal{H} = T + U = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

Clean this up a bit by defining

$$x' \equiv \sqrt{m\omega}\, x \quad \text{and} \quad p' \equiv p/\sqrt{m\omega}$$

then $\dot{X}' = \omega p'$

$$\dot{p}' = -\omega X'$$

$$\frac{d}{dt}\begin{pmatrix} X' \\ p' \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} X' \\ p' \end{pmatrix}$$

uncouple! $\ddot{X}' = \omega \dot{p}' = -\omega^2 X'$

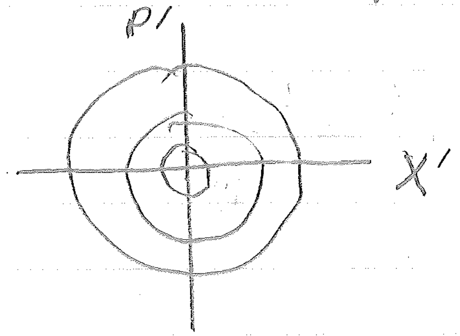
$$X'(t) = X_0' \cos\omega t + P_0' \sin\omega t$$

$$P'(t) = \frac{1}{\omega}\dot{X}' = X_0'(-\sin\omega t) + P_0' \cos\omega t$$

$$\begin{pmatrix} X'(t) \\ P'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}}_{\text{rotation in phase space}}\begin{pmatrix} X_0' \\ P_0' \end{pmatrix}$$

$$X'^2 + p'^2 = X_0^2 + P_0^2 = \text{const}$$

$$E = H = \frac{1}{2}\omega\left(X'^2 + p'^2\right)$$



circles in phase space.

Each circle corresponds to different value of ω.

Example 2: Uniform acceleration

$$H = T + U = \frac{p^2}{2m} - max$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = ma$$

force is $-\vec{\nabla} U$

$$p(t) = mat + p_0$$

$$\dot{x} = at + p_0/m \; ; \; x(t) = \frac{1}{2}at^2 + \frac{p_0}{m}t + x_0$$

consider four trajectories:

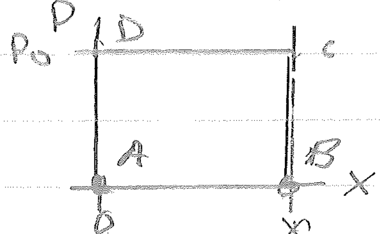
$$\xi_A = (0,0)$$

$$\xi_B = (x_0, 0)$$

$$\xi_C = (x_0, p_0)$$

$$\xi_D = (0, p_0)$$

Volume in phase space (2D space is an area)



$$V = x_0 p_0$$

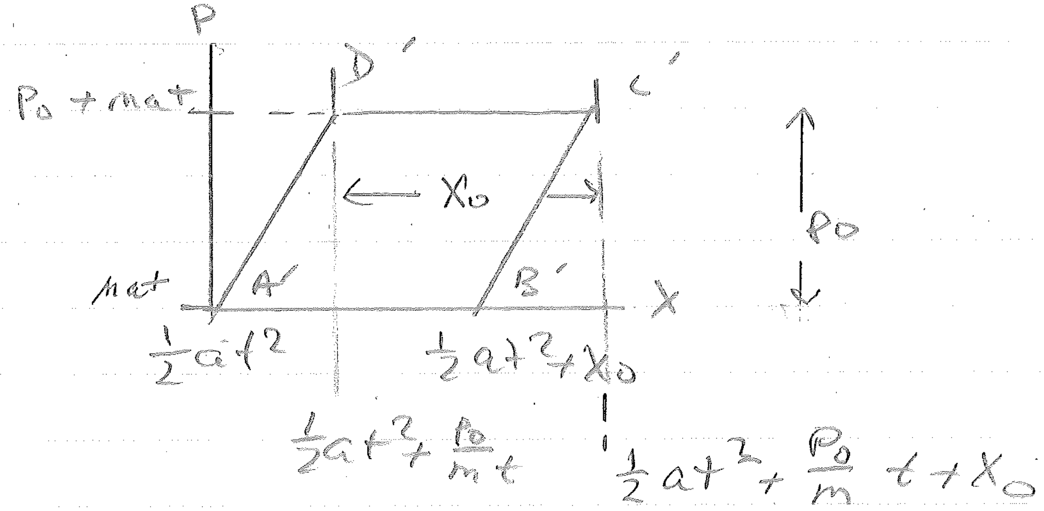
<u>After time t:</u>

$$\xi_A' = \left(\tfrac{1}{2}at^2, mat\right)$$

$$\xi_B' = \left(\tfrac{1}{2}at^2 + X_0, mat\right)$$

$$\xi_C' = \left(\tfrac{1}{2}at^2 + \tfrac{p_0}{m}t + X_0, mat + p_0\right)$$

$$\xi_D' = \left(\tfrac{1}{2}at^2 + \tfrac{p_0}{m}t, mat + p_0\right)$$



V' is parallelogram: $\overrightarrow{A'B'} = X_0 \hat{X}$

$$\overrightarrow{A'D'} = \frac{p_0}{m}t\,\hat{X} + p_0\,\hat{P}$$

$$V' = \left|\overrightarrow{A'B'} \times \overrightarrow{A'D'}\right| = X_0 p_0$$

This is an example of <u>Liouville's Theorem</u>.