Lecture 17: Liouville's Theorem

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1 Phase Space Density

State of system with N degrees of freedom and corresponding generalized coordinates $\{q_i\}$ and generalized momenta s $\{p_i\}$, we define N-tuples $\bar{q} = (q_1, q_2, ...q_N)$ and $\bar{p} = (p_1, p_2, ...p_N)$ and then the phase space has 2N dimensions and a point in this phase space is

$$\bar{\xi} = (\bar{q}, \bar{p})$$

System evolves in time tracing out a curve in phase space. The same system with a different energy will be a family of curves.

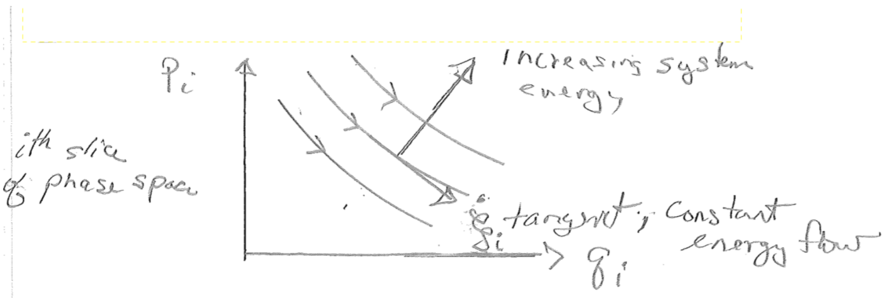


Figure 1: Trajectories in phase space for equivalent systems with different energies.

For a system with a very large number of degrees of freedom, e.g. 1 mole (6×10^{23}) of gas molecules or a galaxy of stars and gas, the dimension of phase space is very, very large– for a monatomic gas, dim. $= 2 \times 3 \times$(#molecules)! Since we cannot practically identify the point in this phase space corresponding to the system at a time t, we instead consider a statistical ensemble of equivalent (representative) systems. We represent the ensemble by a **phase space density (ρ)**.

The flow along a trajectory has zero divergence:

$$\nabla \cdot \dot{\xi} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} = 0$$

Phase space flow along trajectory analogous to incompressible fluid flow along streamline.

Consider a constant temperature gas and representative "slices" of phase space at some instant t.

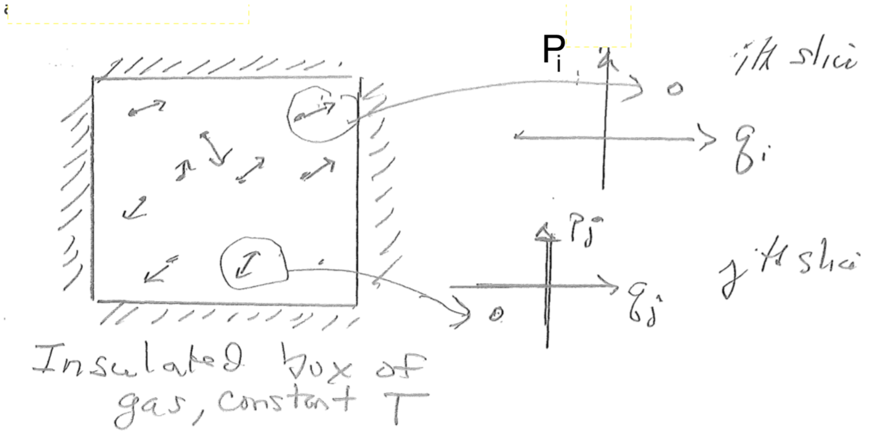


Figure 2: Constant temperature gas molecules. Slices in phase space correspond to the q, p of a particular molecule.

Energy of system is conserved even though energy is exchanged between individual molecules. System's trajectory in phase space follows constant energy path.

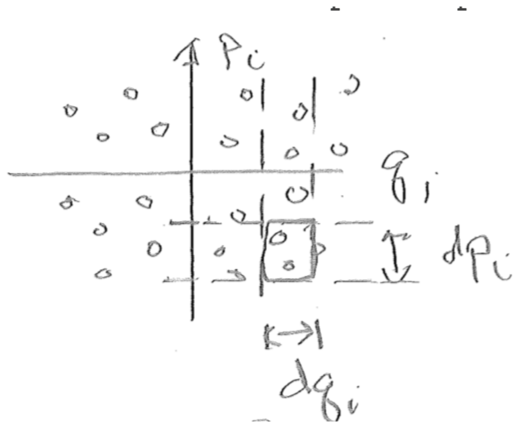


Figure 3: statistical ensemble in phase space showing the i^{th} slice.

The number of ensembles is equal to the integral of the phase space density over the phase space volume.

$$\mathcal{N}_{ensembles} = \int \rho dq^N dp^N$$

2 Proof of Liouville's Theorem (following Marion)

We now consider an arbitrary box in the 2N-dimensional phase space. As the statistical ensemble evolves in time, the points in the ensemble flow in and out of this box. Liouville's Theorem is that the number of points inside this box changes with time due to the flow in minus the flow out. That is, the density of representative points in phase space corresponding to the motion of a system of particles remains constant along the phase space trajectories. (See Figure 1). The total rate of change of time of the density, following a volume in phase space with the flow, is zero:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot \dot{\xi} = 0$$

This is called the material derivative in fluid mechanics, to be discussed next semester. The flow is analogous to that of an incompressible fluid.

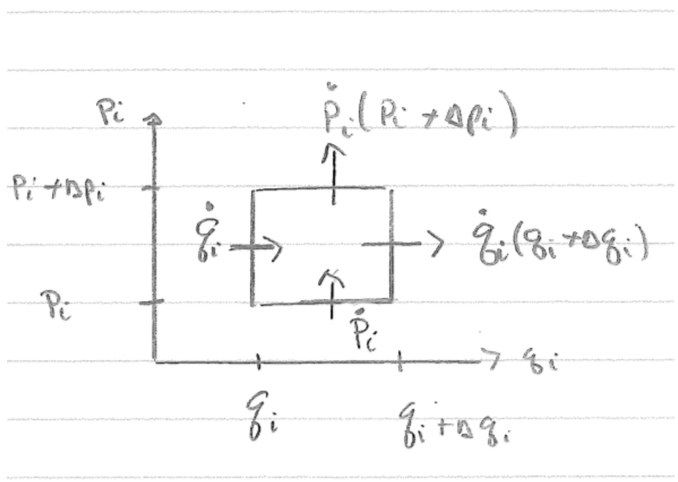


Figure 4: Imaginary box in phase space (i^{th} slice)

Number of phase space points into volume in time Δt in i^{th} slice of phase space:

$$N_{in}^{i} = \rho(q_i, p_i)(\dot{q}_i \Delta p_i + \dot{p}_i \Delta q_i)\Delta t$$

Number of points moving out of i^{th} slice: Taylor expand to first order $\rho, \dot{q}, \dot{p}$ at $q+\Delta q$, $p+\Delta p$

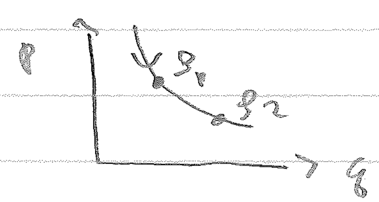
$$N_{out}^{i} = \left(\rho + \frac{\partial \rho}{\partial q_i}\Delta q_i + \frac{\partial \rho}{\partial p_i}\Delta p_i\right)\left\{\left(\dot{q}_i + \frac{\partial \dot{q}_i}{\partial q_i}\Delta q_i\right)\Delta p_i + \left(\dot{p}_i + \frac{\partial \dot{p}_i}{\partial p_i}\Delta p_i\right)\Delta q_i\right\}\Delta t$$

$$= N_{in}^{i} + \rho\underbrace{\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right)}_{\text{zero by Hamilton's eq.}}\Delta p_i \Delta q_i \Delta t + \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)\Delta p_i \Delta q_i \Delta t$$

$$N_{in}^{i} - N_{out}^{i} = \frac{\partial \rho}{\partial t}\Delta q_i \Delta p_i \Delta t = -\left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)\Delta q_i \Delta p_i \Delta t$$

divide out Δ's and sum to get total

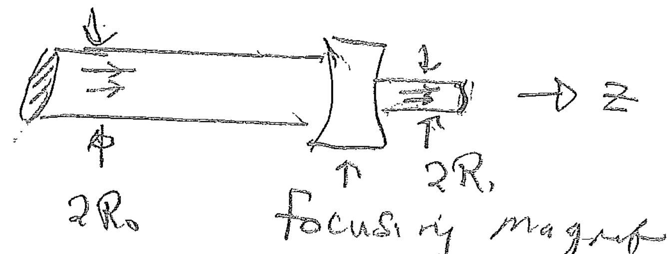
$$\frac{\partial \rho}{\partial t} + \sum_i \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) = \frac{d\rho}{dt} = 0$$



$$\frac{d\rho}{dt} = \lim_{t_2 - t_1 \to 0} \frac{\rho(q_2, p_2, t_2) - \rho(q_1, p_1, t_1)}{t_2 - t_1}$$

Density in phase space along phase space trajectory is constant.

Example: focusing charged particle beam



R_0, R_1 beam radii before, after magnet

Assume constant phase space density across beam pipe and in transverse (to beam) momentum up to some max $P_\perp$.

Volume in phase space, (ρ = constant = 1)

$$V_0 = \left(\int_0^{R_0} r\,dr \int_0^{2\pi} d\phi\right)\left(\int_0^{P_\perp^0} p\,dp \int_0^{2\pi} d\phi_p\right)$$

$$= \pi^2 R_0^2 P_\perp^{0\,2}$$

similarly,

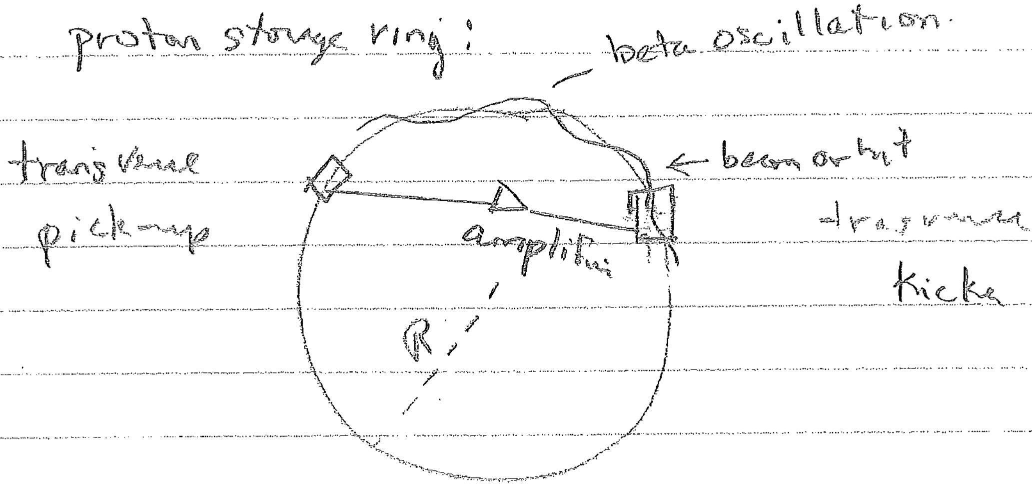
$$V_1 = \pi^2 R_1^2 P_\perp^{1\,2}$$

$$V_0 = V_1 \Rightarrow P_\perp^1 = \left(\frac{R_0}{R_1}\right) P_\perp^0$$

focused beam has greater divergence

$\boxed{\text{Stochastic Cooling}}$ (Nobel, 1989 Simon van der Meer)

proton storage ring:



Stochastic Cooling defeat's Liouville!

review article, see arXiv:physics/0308044

original 1972 paper http://cds.cern.ch/record/312939/files/

"As is well known, Liouville's theorem predicts that betatron oscillations cannot be damped by the use of electromagnetic fields deflecting the particles. However, this theorem is based on statistics and is only strictly valid either for an infinite number of particles, or for a finite number if no information is available about the position in phase plane of the individual particles. Clearly, if each particle could be separately observed and a correction applied to its orbit, the oscillations could be suppressed."