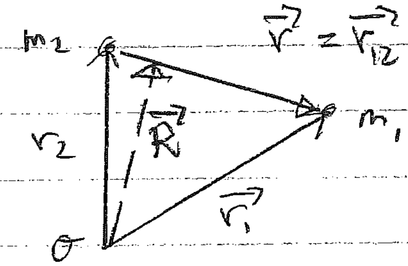
Lecture # 1 : Two Body Central Force

Taylor, Ch. 8



$\vec{r} = \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ relative separation

Central force: $U(\vec{r}_1, \vec{r}_2) = U(\vec{r}_1 - \vec{r}_2) = U(|\vec{r}|)$

e.g. Newtonian Gravity

$$U = -\frac{G m_1 m_2}{r_{12}}$$

spherically symmetric potential

For isolated 2-body system (no external force) total momentum is conserved.

$$\vec{P}_1 + \vec{P}_2 \equiv \vec{P}_{cm} = \text{constant}$$

$$m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \vec{P}_{cm} = (m_1 + m_2) \dot{\vec{R}}_{cm}$$

where $\vec{R}_{cm} \equiv \left(\frac{1}{m_1 + m_2}\right)(m_1 \vec{r}_1 + m_2 \vec{r}_2)$

and $\vec{r}_1 = \vec{R}_{cm} + \frac{m_2}{m_1 + m_2} \vec{r}$

$$\vec{r}_2 = \vec{R}_{cm} - \frac{m_1}{m_1 + m_2} \vec{r}$$

Center of Momentum (mass) frame - inertial frame (CM) in which total momentum is zero.

Reduced Mass Rewrite kinetic energy
($m_T \equiv m_1 + m_2$)

$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 = \frac{1}{2} m_1 \left(\dot{R} + \frac{m_2}{m_T} \dot{r}\right)^2$$

$$+ \frac{1}{2} m_2 \left(\dot{R} - \frac{m_1}{m_T} \dot{r}\right)^2$$

$$= \frac{1}{2} m_T \dot{R}^2 + \frac{1}{2} \frac{m_1 m_2}{m_T^2} (m_1 + m_2) \dot{r}^2$$

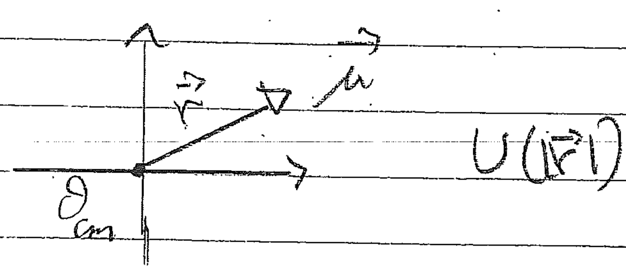
$$= \frac{1}{2} m_T \dot{R}^2 + \frac{1}{2} \left(\frac{m_1 m_2}{m_T}\right) \dot{r}^2$$

reduced mass $\mu \equiv m_1 m_2 / (m_1 + m_2)$

$$\mathcal{L} = T - U = \frac{1}{2} m_T \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - U(r)$$

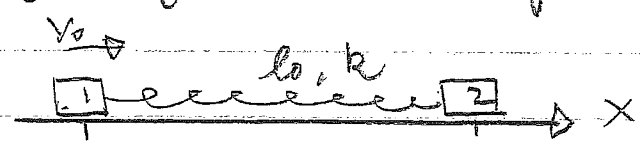
R) eq. $\ddot{R} = 0$; $\dot{R} = \text{Const} = V_{cm} = \frac{P_{cm}}{m_T}$

In cm frame ($P_{cm} = 0$), we have equivalent problem of fictitious reduced mass particle moving around fixed force center.



$$p = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \dot{r} \quad \text{canonical momentum}$$

<u>1D example</u> : 2 masses connected by spring sliding on frictionless surface!



$X_1(0) = 0$ $\qquad$ $X_2(0) = l_0$

$\dot{X}_1(0) = V_0$

$$r = X_1 - X_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$R = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad U(r) = \frac{1}{2} k (r - l_0)^2$$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\mu \dot{r}^2 - U(r)$$

$$\ddot{R} = 0 \qquad \dot{R} = \text{const} = \left(\frac{m_1}{m_1 + m_2}\right) V_0$$

$$R(t) = \left(\frac{m_1}{m_1 + m_2}\right) V_0 t + \frac{m_2 l_0}{m_1 + m_2}$$

$$\mu \ddot{r} = -k(r - l_0) \qquad \text{let } y = r - l_0, \quad \omega = \sqrt{\frac{k}{\mu}}$$

$$r(0) = -l_0$$
$$\dot{r}(0) = V_0$$

$$r(t) = -l_0 + \frac{V_0}{\omega} \sin \omega t$$

$$X_1(t) = R(t) + \frac{m_2}{m_1 + m_2} r(t)$$
$$X_2(t) = R(t) - \frac{m_1}{m_1 + m_2} r(t)$$

So:

$$x_1(t) = \frac{1}{m_1+m_2}\left[m_2 l_0 + m_1 V_0 t - m_2 l_0 + \frac{m_2 V_0}{\omega}\sin\omega t\right]$$

$$= \frac{V_0}{m_1+m_2}\left[m_1 t + \frac{m_2}{\omega}\sin\omega t\right]$$

$$x_2(t) = \frac{1}{m_1+m_2}\left[m_2 l_0 + m_1 V_0 t + m_1 l_0 - \frac{m_1 V_0}{\omega}\sin\omega t\right]$$

$$= l_0 + \frac{V_0}{m_1+m_2}\left[m_1 t - \frac{m_1}{\omega}\sin\omega t\right]$$

<u>Conservation of $\vec{L}$</u> (orbital angular momentum)

$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$ in CM frame $\vec{R} = 0$,

$\vec{r}_1 = \frac{m_2}{m_T}\vec{r}$; $\vec{r}_2 = -\frac{m_1}{m_T}\vec{r}$

$$\vec{L} = m_1\left(\frac{m_2}{m_T}\right)^2 \vec{r}\times\dot{\vec{r}} + m_2\left(\frac{m_1}{m_T}\right)^2 \vec{r}\times\dot{\vec{r}} = \frac{m_1 m_2}{m_T}\vec{r}\times\dot{\vec{r}}$$
$$= \mu\vec{r}\times\dot{\vec{r}} = \vec{r}\times\vec{p}$$

For central force $\vec{F} = -\vec{\nabla}U = -\hat{r}\frac{\partial}{\partial r}U = -\hat{r}U'$

so $\vec{r}\times\vec{F} = 0 = \vec{r}\times\frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{L}}{dt}$

since $\dot{\vec{r}}\times(\mu\dot{\vec{r}}) = 0$

<u>Fancy proof</u>: recall Poisson bracket.

$$\frac{d\vec{L}}{dt} = [\vec{L}, H] = [\vec{L}, T] + [\vec{L}, U]$$

$$L_3 = X_1 P_2 - X_2 P_1$$

$$[L_3, T] = \sum_{i=1}^{3} \left[\frac{\partial L_3}{\partial x_i} \frac{\partial T}{\partial p_i} - \frac{\partial L_3}{\partial p_i} \overset{0}{\frac{\partial T}{\partial x_i}} \right]$$

$$= \frac{\partial L_3}{\partial x_1} \frac{\partial T}{\partial p_1} + \frac{\partial L_3}{\partial x_2} \frac{\partial T}{\partial p_2} = p_2 \left(\frac{p_1}{\mu}\right) - p_1 \left(\frac{p_2}{\mu}\right) = 0$$

$$[L_3, U] = \sum_{i=1}^{3} \left[\frac{\partial L_3}{\partial x_i} \overset{0}{\frac{\partial U}{\partial p_i}} - \frac{\partial L_3}{\partial p_i} \frac{\partial U}{\partial x_i} \right]$$

$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial x_i} = \frac{\partial U}{\partial r} \frac{x_i}{r}$$

$$[L_3, U] = X_2 \frac{\partial U}{\partial r} \frac{x_1}{r} - X_1 \frac{\partial U}{\partial r} \frac{x_2}{r} = 0$$

<u>Moreover</u>, from spherical symmetry of $U(|\vec{r}|)$ it follows that $\frac{d}{dt}(\vec{L}) = 0$. (rotational invariance)

Since direction of $\vec{L}$ is fixed, we have motion in a plane.

$$\vec{r} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

and $v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

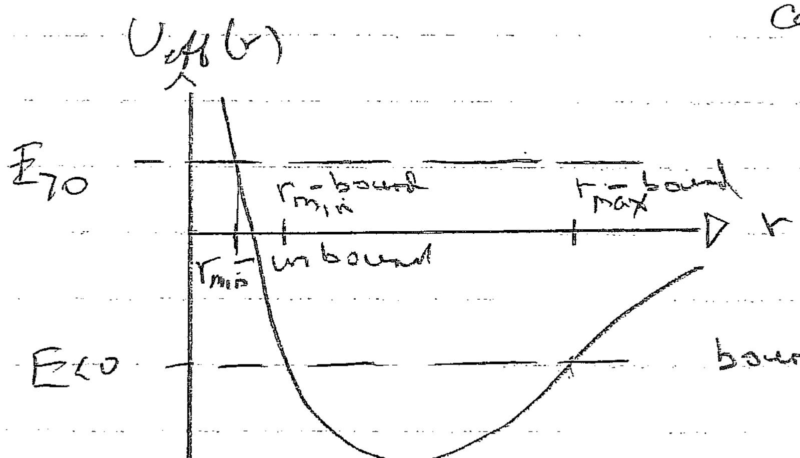
ϕ is cyclic $\quad L_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{constant} \equiv l$

Orbital angular momentum L_ϕ is conserved.

<u>r equation</u>.

$$\mu\ddot{r} = \mu r\dot{\phi}^2 - U' = \frac{l^2}{\mu r^3} - U' \equiv -\frac{d}{dr}U_{eff}$$

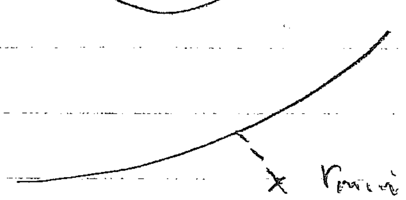
effective potential $U_{eff}(r) = \underbrace{\frac{l^2}{2\mu r^2}}_{\text{centrifugal barrier}} + U(r)$



unbound orbit $E>0$

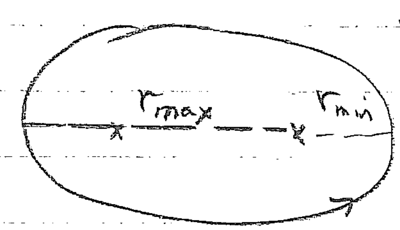
bound orbit $E<0$

$r(\phi)$



unbound hyperbolic

$r(\phi)$



bound, ellipse