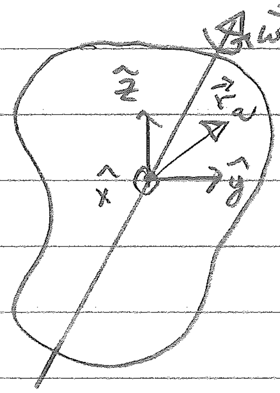
Physics 304 Spring '2018

Lecture #10: Principal Axes

Origin of coordinates on axis of rotation.



$$\vec{L}_{rot} = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a = \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a)$$

with $\vec{A} \times (\vec{B} \times \vec{A}) = A^2 \vec{B} - \vec{A}(\vec{A} \cdot \vec{B})$ identity

$$\vec{L} = \sum_a m_a \left[r_a^2 \vec{\omega} - \vec{r}_a (\vec{\omega} \cdot \vec{r}_a) \right]$$

$$L_i = \sum_a m_a \left[r_a^2 \omega_i - r_{a_i} \sum_j \omega_j r_{a_j} \right]$$

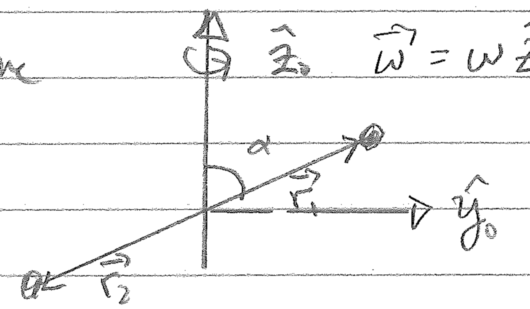
$$= \sum_j \omega_j \left[\sum_a m_a \left(r_a^2 \delta_{ij} - r_{a_i} r_{a_j} \right) \right]$$

$$= \sum_j I_{ij} \omega_j$$

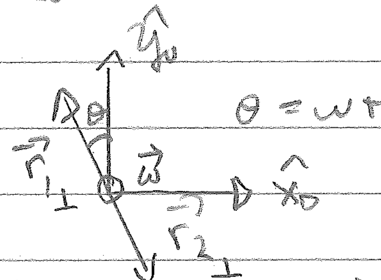
$\vec{L} = \bar{I} \cdot \vec{\omega}$ $\qquad$ $\vec{L}, \vec{\omega}$ not necessarily parallel

Example: rotated dumbell (2 masses m connected by rigid, massless rod of length ℓ)

Inertial frame at $t=0$



looking down $\hat{z}_0$ axis:



$$\vec{r}_{1_\perp} = \frac{\ell}{2} S_\alpha (\cos\omega t \, \hat{y}_0 - \sin\omega t \, \hat{x}_0) \qquad S_\alpha \equiv \sin\alpha$$

$$\vec{r}_{2_\perp} = \frac{\ell}{2} S_\alpha (-\cos\omega t \, \hat{y}_0 + \sin\omega t \, \hat{x}_0)$$

$$\vec{L} = \sum_{a=1}^{2} m \, \vec{r}_a \times \dot{\vec{r}}_a \qquad (\text{with } S_t = \sin\omega t, \; C_t = \cos\omega t)$$

$$\vec{r}_1 \times \dot{\vec{r}}_1 = \frac{\ell^2 \omega}{4} \left(S_\alpha C_t \hat{y}_0 - S_\alpha S_t \hat{x}_0 + C_\alpha \hat{z}_0 \right) \times$$

$$\left(-S_\alpha S_t \hat{y}_0 - S_\alpha C_t \hat{x}_0 \right)$$

$$= \frac{\ell^2 \omega}{4} \left[+S_\alpha^2 C_t^2 \hat{z}_0 + S_\alpha^2 S_t^2 \hat{z}_0 + S_\alpha C_\alpha S_t \hat{x}_0 - S_\alpha C_\alpha C_t \hat{y}_0 \right]$$

$$= \frac{\ell^2 \omega}{4} \left[S_\alpha^2 \hat{z}_0 + S_\alpha C_\alpha (S_t \hat{x}_0 - C_t \hat{y}_0) \right]$$

2nd mass gives same, so

$$L_{z_0} = \left(\frac{m\omega l^2}{2}\right) S_\alpha^2$$

$$L_{x_0} = \left(\frac{m\omega l^2}{2}\right) S_\alpha C_\alpha \sin\omega t$$

$$L_{y_0} = \left(\frac{m\omega l^2}{2}\right) S_\alpha C_\alpha (-\cos\omega t)$$

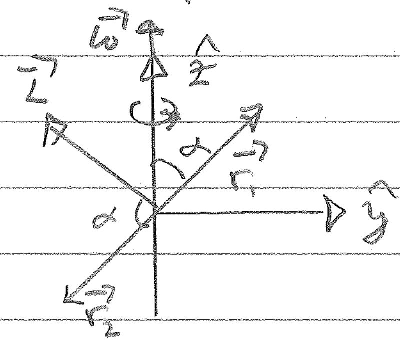
} rotation about z-axis

$$\begin{pmatrix} L_{x_0} \\ L_{y_0} \\ L_{z_0} \end{pmatrix}_{\text{inertial}} = \begin{pmatrix} C_t & -S_t & 0 \\ S_t & C_t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -C_\alpha \\ S_\alpha \end{pmatrix}_{\text{body}} \left(\frac{S_\alpha m\omega l^2}{2}\right)$$

ωt

Right-handed rotation of coordinates

Body frame: axes correspond between frames @ $t=0$



$$\vec{r}_1 = \frac{l}{2}\left(S_\alpha \hat{y} + C_\alpha \hat{z}\right)$$

$$\vec{V}_1 = \vec{\omega} \times \vec{r}_1 = \frac{l\omega}{2} S_\alpha (\hat{z} \times \hat{y}) = -\frac{l\omega}{2} S_\alpha \hat{x}$$

$$\vec{L} = 2m\, \vec{r}_1 \times \dot{\vec{r}}_1 = 2m\left(\frac{l}{2}\right)^2 \omega (-S_\alpha)\left(S_\alpha\, \hat{y}\times\hat{x} + C_\alpha\, \hat{z}\times\hat{x}\right)$$

$$= \frac{m l^2 \omega}{2} S_\alpha\left(-C_\alpha \hat{y} + S_\alpha \hat{z}\right)$$

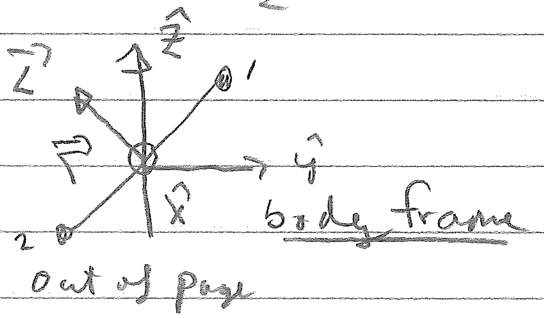
$$\vec{L} \xrightarrow[\text{body}]{} \begin{pmatrix} 0 \\ -C_\alpha \\ S_\alpha \end{pmatrix}_{\text{body}} \left(\frac{S_\alpha m\omega l^2}{2}\right)$$

Torque $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \vec{\omega} \times \vec{L}$

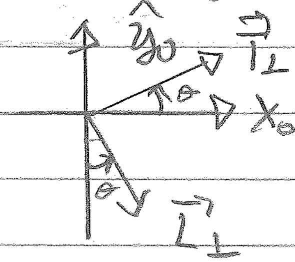
$$= \frac{m\omega^3 l^2}{2} S_\alpha C_\alpha \left[\cos\omega t \; \hat{x}_0 + \sin\omega t \; \hat{y}_0\right]$$

$$= \omega\hat{z} \times \left[\frac{m l^2 \omega}{2} S_\alpha \left(-C_\alpha \hat{y} + S_\alpha \hat{z}\right)\right]$$

$$= \frac{m\omega^2 l^2}{2} S_\alpha C_\alpha \; \hat{x}$$

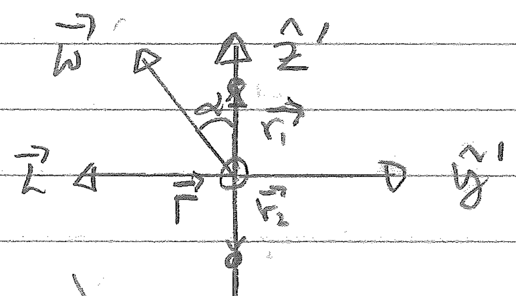


Looking down $\hat{z}_0$ axis in inertial frame:



$\theta = \omega t$

No reason to choose body frame with $\hat{z} \parallel \hat{z}_0$. Much simpler to make $\bar{I}$ diagonal.



dumbbell along $\hat{z}'$ axis

$$\vec{\omega} = \begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}_{'\text{basis}} \qquad \bar{I}' = \frac{ml^2}{2} \operatorname{diag}(1,1,0)$$

$$\vec{L} = \bar{I}\cdot\vec{\omega} = \omega \frac{ml^2}{2} S_\alpha (-\hat{y}')$$

$$\vec{\Gamma} = \vec{\omega} \times \vec{L} = \omega^2 \frac{ml^2}{2} S_\alpha C_\alpha \; \hat{x}'$$

Principal Axes Euler, 1750

For any rigid body, $\bar{I}$ can be made diagonal by suitable choices of axes.
(Math- real, symmetric matrix)

$$\bar{\bar{I}} = \mathrm{diag}(I_1, I_2, I_3) \quad \text{(single subscript)}$$
or λ_i

I_i called principal moments
axes called principal axes.

for $\vec{\omega}$ along principal axes, say $\vec{\omega} = (\omega, 0, 0)$

$$\vec{L} = I_1 \vec{\omega}, \quad \vec{\omega} \times \vec{L} = 0 \quad \text{Torque is zero}$$
$\vec{L}$ fixed direction in inertial frame

Finding Principal Axes (diagonalize matrix)

3 Principal axes $\hat{\omega}_p$, $\vec{L} = \bar{\bar{I}} \cdot \hat{\omega}_p = I_p \hat{\omega}_p$
or
$$\left(\bar{\bar{I}} - I_p \bar{\bar{1}}\right) \cdot \hat{\omega}_p = 0$$
$\uparrow$ 3x3 unit matrix $(\bar{\bar{1}})_{ij} = \delta_{ij}$

Solve characteristic (eigenvalue) equation

$$\det \left| I_{ij} - I_p \delta_{ij} \right| = 0$$
Cubic, 3 real roots (see Appendix)

Once I_p are known,

$$\left(\bar{\bar{I}} - I_p \bar{\bar{1}}\right) \cdot \hat{\omega}_p = 0$$

gives 2 unique equations (since $|\hat{\omega}_p|^2 = 1$)

<u>Relation to rotation Matrices</u>

Find rotation $\bar{\bar{R}}$ that diagonalizes $\bar{\bar{I}}$.

$$\bar{\bar{I}}' = \bar{\bar{R}} \cdot \bar{\bar{I}} \cdot \bar{\bar{R}}^{-1} \quad \text{or} \quad \bar{\bar{I}}' \cdot \bar{\bar{R}} = \bar{\bar{R}} \cdot \bar{\bar{I}}$$

$$\bar{\bar{R}} \cdot \bar{\bar{I}} - \bar{\bar{I}}' \bar{\bar{R}} = 0 \qquad (*)$$

$$\left(\bar{\bar{R}} \cdot \bar{\bar{I}}\right)_{ij} = \sum_k R_{ik} I_{kj}$$

$$\left(\bar{\bar{I}}' \bar{\bar{R}}\right)_{ij} = \sum_k I'_i \delta_{ik} R_{kj} = I'_i R_{ij} = \sum_k I'_i \delta_{kj} R_{ik}$$

So in component form eq. $(*)$ is

$$\sum_k \left(I_{kj} - I'_i \delta_{kj}\right) R_{ik} = 0$$

same secular equation with $\underset{\substack{\uparrow \\ \text{row index}}}{R_{ik}} = \left(\hat{\omega}_i\right)_k$

$\hat{\omega}_p$ vectors as rows of matrix

$$\bar{\bar{R}} = \begin{pmatrix} \omega_{1x} & \omega_{1y} & \omega_{1z} \\ \omega_{2x} & \omega_{2y} & \omega_{2z} \\ \omega_{3x} & \omega_{3y} & \omega_{3z} \end{pmatrix}$$

In practice, rarely need to use this rotation matrix

<u>Example</u>: thin plate "lamina"
from Latin "thin sheet of material"

On homework, general form of lamina is

$$\overline{\overline{I}}_{\text{lamina}} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

take $\overline{\overline{I}} = A \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$

2x2 eigenvalue equation

$$A \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \hat{\omega} = \lambda \hat{\omega}$$

$$\det \begin{vmatrix} 1-\lambda, & -\frac{1}{2} \\ -\frac{1}{2}, & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - \frac{1}{4} = 0$$
$$1-\lambda = \pm \frac{1}{2}$$
$$\lambda = \frac{1}{2}, \ 3/2$$

$\lambda = \frac{1}{2}$ $\begin{pmatrix} 1-\frac{1}{2}, & -\frac{1}{2} \\ -\frac{1}{2}, & 1-1/2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$

$\frac{1}{2} w_1 - \frac{1}{2} w_2 = 0$ (2nd eq. redundant)

$w_1 = w_2$

$\hat{w}_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 3/2$ $\begin{pmatrix} 1-3/2, & -\frac{1}{2} \\ -\frac{1}{2}, & 1-3/2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$

$\hat{w}_{3/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

note $\hat{w}_{\frac{1}{2}} \cdot \hat{w}_{3/2} = 0$

<u>check</u> $\bar{I}' = \bar{R}\,\bar{I}\,\bar{R}^{-1}$

$$\bar{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{array}{l} \lambda = 3/2 \text{ row} \\ \lambda = \frac{1}{2} \text{ row} \end{array}$$

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3/2 & 1/2 \\ -3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$