Physics 304 Spring 2018

Lecture #11 Euler's Equations

torque $\vec{\Gamma} = \frac{d\vec{L}}{dt}$

Components in body frame,

$$\Gamma_i = \dot{L}_i + \sum_{j,k} \varepsilon_{ijk} \omega_j L_k$$

take body frame to make $\overline{I} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$L_i = \lambda_i \omega_i$$

$$\Gamma_i = \lambda_i \dot{\omega}_i + \sum_{j,k} \varepsilon_{ijk} \omega_j \omega_k \lambda_k$$

Euler's equations

$$\Gamma_1 = \lambda_1 \dot{\omega}_1 + \omega_2 \omega_3 \lambda_3 - \omega_3 \omega_2 \lambda_2$$

$$\Gamma_1 = \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2 \omega_3$$ (easier to remember)

and cyclic: $(2,3,1)$, $(3,1,2)$

Torque free motion $\boxed{\vec{\omega} = \omega_3 \hat{e}_3 \;\; \dot{\omega}_3 = 0}$

$\vec{L} = \lambda_3 \omega_3 \vec{e}_3$ ~~so~~ $\vec{\omega} \times \vec{L} = 0$

and $\vec{\Gamma} = 0$. $\vec{L}, \vec{\omega}$ aligned and fixed in inertial frame.

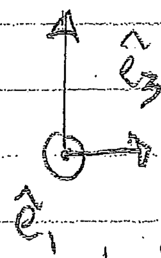
More generally, $\vec{\omega}$ not along principal axis.
Torque free,

$$\mathring{\vec{L}} + \vec{\omega}\times\vec{L} = 0 \qquad \text{(body frame components)}$$

$$\lambda_i \mathring{\omega}_i + \sum_{j,k} \varepsilon_{ijk}\,\lambda_k\,\omega_j\,\omega_k = 0$$

ω_i time dependent (body frame components)

<u>Stability</u> Tennis Racket Theorem



$\hat{e}_3$ (handle)

$\hat{e}_2$ in plane of face

$\hat{e}_1$ $\perp$ to face $\qquad \lambda_3 < \lambda_2 < \lambda_1$

Torque free motion with $\vec{\omega}(0) = \omega_0 \hat{e}_3$
We have approximately, $\omega_1 \ll \omega_3$ & $\omega_2 \ll \omega_3$

$$\lambda_3 \mathring{\omega}_3 = (\lambda_1 - \lambda_2)\,\omega_2\,\omega_1 \approx 0 \qquad \omega_3(t) \approx \omega_0$$

$$\left.\begin{aligned} \lambda_1 \mathring{\omega}_1 &= (\lambda_2 - \lambda_3)\,\omega_2\,\omega_3 \\ \lambda_2 \mathring{\omega}_2 &= (\lambda_3 - \lambda_1)\,\omega_3\,\omega_1 \end{aligned}\right] \text{ two coupled}$$

$$\lambda_1 \ddot{\omega}_1 \approx (\lambda_2 - \lambda_3)\,\omega_3\,\mathring{\omega}_2 = \frac{(\lambda_2 - \lambda_3)}{\lambda_2}(\lambda_3 - \lambda_1)\,\omega_3^2\,\omega_1$$

$$\ddot{\omega}_1 = -\left[\frac{\omega_0^2\,(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\right]\omega_1$$

$$[\;] = \Omega_3^2 > 0$$

and $\ddot{\omega}_2 = -\Omega_3^2 \omega_2$

both equations have harmonic oscillation solutions.
So rotating about $\hat{e}_3$ gives <u>stable</u> motion.

Suppose we take initial $\vec{\omega}(0) = \omega_0 \hat{e}_1$. We can simply permute the indices to get

$$\Omega_1^2 = \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 \lambda_2} \omega_0^2 > 0$$

Also, stable harmonic oscillations. However, for $\vec{\omega}_0(0) = \omega_0 \hat{e}_2$,

$$-\Omega_2^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}{\lambda_3 \lambda_1} \omega_0^2 < 0$$

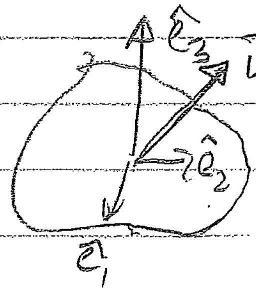
giving $\ddot{\omega}_{1,3} = +\Omega_2^2 \omega_{1,3}$

with exponentially growing solutions.

For intermediate eigenvalue $\underline{\lambda_2}$, rotation about $\hat{e}_2$ axis is unstable

Example 10.41 torque free lamina

lamina in $\hat{e}_1, \hat{e}_2$ plane



$$\bar{\bar{I}} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

We can show $\frac{d}{dt}(\vec{\omega}_\perp) = 0 = \dot{\omega}_1 \hat{e}_1 + \dot{\omega}_2 \hat{e}_2$

Diagonalize $\bar{\bar{I}} = \text{diag}(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$

$$\lambda_{1,2} = \frac{A+B}{2} \pm \sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}$$

$$\lambda_1 + \lambda_2 = A + B$$

and $\frac{A+B}{2} > \sqrt{\left(\frac{A-B}{2}\right)^2 + C^2}$

$$(A+B)^2 > (A-B)^2 + 4C^2$$

$$\boxed{AB > C^2}$$

Euler's equations:

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3 = -\lambda_1 \omega_2 \omega_3$$
$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_1\omega_3 = +\lambda_2 \omega_1 \omega_3$$

$$\frac{\dot{\omega}_1}{\dot{\omega}_2} = -\frac{\omega_2}{\omega_1} \qquad \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$$

so $\vec{\omega}_\perp \cdot \frac{d\vec{\omega}_\perp}{dt} = 0$ and $\frac{d\omega_\perp^2}{dt} = 0$

Torque-Free Top

top: $\lambda_1 = \lambda_2 \neq \lambda_3$

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 = 0$$

$$\boxed{\omega_3 \text{ is constant}}$$

Other 2 Euler equations are

$$\dot{\omega}_1 = \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right) \omega_2 \omega_3 \equiv \Omega_b \omega_2$$

$$\dot{\omega}_2 = \frac{(\lambda_3 - \lambda_1)}{\lambda_1} \omega_3 \omega_1 \equiv -\Omega_b \omega_1$$

b for body $\quad \Omega_b \equiv \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right) \omega_3 \quad > 0$ for football

Note two cases:

prolate (football) $\quad \lambda_3 < \lambda_1 \quad \Omega_b > 0$
oblate (plate) $\quad \lambda_3 > \lambda_1 \quad \Omega_b < 0$

Solve coupled equations by $\eta \equiv \omega_1 + i\omega_2$

or $\quad \ddot{\omega}_1 = -\Omega_b^2 \omega_1, \quad \ddot{\omega}_2 = -\Omega_b^2 \omega_2$

Solution $\quad \eta = \eta_0 e^{-i\Omega_b t}$

$$\eta(t) = \omega_1(t) + i\,\omega_2(t) =$$

$$(\omega_1^0 + i\omega_2^0)(\cos\Omega_b t - i\sin\Omega_b t)$$

$\omega_1^0 \equiv \omega_1(0)$, $\omega_2^0 \equiv \omega_2(0)$

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\Omega_b t\,, & \sin\Omega_b t \\ -\sin\Omega_b t\,, & \cos\Omega_b t \end{pmatrix}}_{\text{rotation about } \hat{e}_3} \begin{pmatrix} \omega_1^0 \\ \omega_2^0 \end{pmatrix}$$

$\vec{\Omega}_b = -\Omega_b = -\left(\frac{\lambda_1-\lambda_3}{\lambda_1}\right)\hat{e}_3$ left-handed ($\Omega_b > 0$) prolate

$\vec{\Omega}_b = -\Omega_b\hat{e}_3 = +\left(\frac{\lambda_3-\lambda_1}{\lambda_1}\right)\hat{e}_3$ right-handed ($\Omega_b < 0$) oblate

Suppose $\omega_2(0) = \omega_0$, $\omega_1(0) = 0$

$\omega_1(t) = \omega_0 \sin\Omega_b t$

$\omega_2(t) = \omega_0 \cos\Omega_b t \qquad \Omega_b = \omega_3\left(\frac{\lambda_1-\lambda_3}{\lambda_1}\right)$

$$\vec{L} = \lambda_1\omega_0(\sin\Omega_b t\,\hat{e}_1 + \cos\Omega_b t\,\hat{e}_2) + \lambda_3\omega_3\hat{e}_3$$

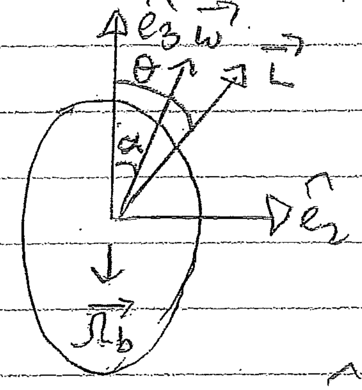
$$\vec{L}\cdot\vec{\omega} = \lambda_1\omega_0^2 + \lambda_3\omega_3^2 = \text{constant}$$

$$(\vec{\omega}\times\vec{L})\cdot\hat{e}_3 = \omega_1 L_2 - \omega_2 L_1 = 0$$

Torque free top, prolate case

sketch at $t = 0$

body frame:



$$\tan\alpha = \frac{\omega_0}{\omega_3}$$

$$\tan\theta = \frac{\lambda_1 \omega_0}{\lambda_3 \omega_3}$$

$$\boxed{\tan\theta = \frac{\lambda_1}{\lambda_3} \tan\alpha}$$

$\vec{\omega}, \vec{L}$ both precess about $\hat{e}_3$

prolate case $\theta > \alpha$.

oblate, direction of $\vec{\Omega}_b$ reversed and $\theta < \alpha$

In inertial frame, $\left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = 0$

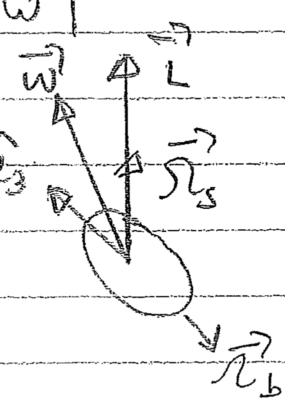
In inertial "space" frame, body rotates with angular velocity

$$\vec{\Omega}_s = \frac{\vec{L}}{\lambda_1} = \vec{\Omega}_b + \vec{\omega}$$

(Need: Euler angles. See HW 10.55)

note $|\vec{\Omega}_b| < |\vec{\omega}|$

space frame:



right-handed rotation about $\vec{L}$.

Chandler Wobble The earth is an oblate top ($\lambda_3 > \lambda_1$)

$\vec{\omega}$ not along body symmetry axis $\hat{e}_3$. undergoes free precession.

We follow R. Baierlein Newtonian Dynamics and model earth as sphere + ring

$$\overline{\overline{I}}^{\,sphere} = \frac{2}{5} M_e R^2 \, \mathrm{diag}(1,1,1)$$

$$\overline{\overline{I}}^{\,ring} = M_e R^2 \, \mathrm{diag}\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \varepsilon\right) \qquad \varepsilon \equiv \frac{m_r}{M_e}$$

$$\frac{\lambda_3}{\lambda_1} = \frac{\frac{2}{5}+\varepsilon}{\frac{2}{5}+\varepsilon/2} \simeq 1 + \left(\frac{5}{2} - \frac{5}{4}\right)\varepsilon = 1 + \frac{5}{4}\varepsilon$$

From sattelite data $\frac{\lambda_3}{\lambda_1} = 1 + \frac{1}{306}$

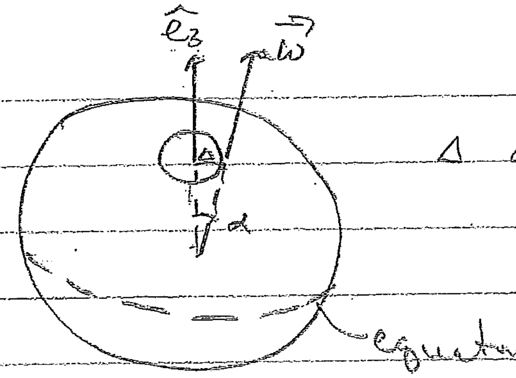
$$\Omega_b = \left(\frac{\lambda_3}{\lambda_1} - 1\right)\omega_3 = \left(\frac{1}{306}\right)\frac{2\pi}{Day} \Rightarrow \boxed{T_p = 306\,d}$$

Chandler (1891) measured 427d

Current 433d

Earth not perfectly rigid (e.g., ocean flows)

Sketch:



Δ measured $\cong$ 10 m $\qquad \alpha = \frac{\Delta}{R} = \frac{10m}{6371 km} = \frac{1}{6} \times 10^{-5}$

$\tan\theta = \frac{\lambda_1}{\lambda_3}\tan\alpha \qquad \alpha \ll 1 \quad (\alpha - \theta) \quad \angle(\vec{L}, \vec{\omega})$

$$\alpha - \theta = \alpha\left(1 - \frac{\lambda_1}{\lambda_3}\right) = \alpha\left(1 - \frac{1}{1 + \frac{1}{305}}\right) \cong \frac{\alpha}{306}$$

Or distance at surface $\delta(\vec{L}, \vec{\omega}) = \frac{\Delta}{306} = \underline{\underline{3\ cm}}$

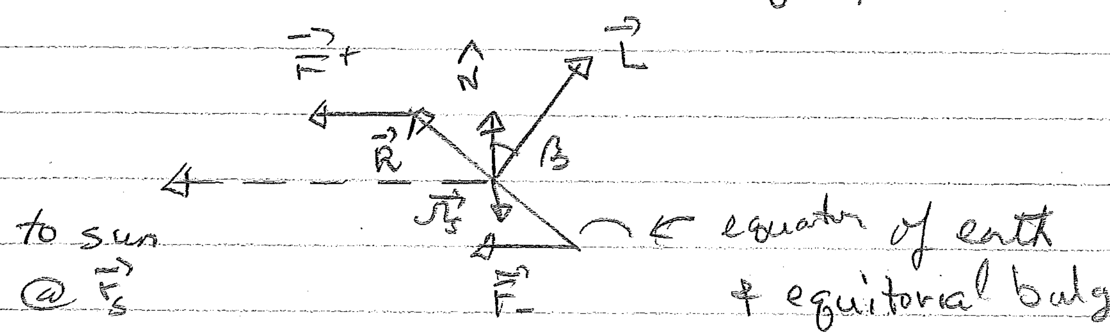
So $\vec{\omega} \approx$ along $\vec{L}$, $\vec{\omega} \approx$ fixed in space
$\vec{L}$ points to North star Polaris.

Δ actually varies irregularly (Chandler wobble)
to factor 2 due to non-rigidity of earth.

Over long period of time $\vec{L}$ direction is not fixed due to gravitational force $1/r^2$ dependence.

Discovered by Greek astronomer Hipparchus ~200 B.C. who compared his observations to Babylonian astronomer Kidinnu (~340 B.C.)

In plane of earth's & circular orbit with normal $\hat{n}$ $\vec{L}$ makes an angle $\beta \simeq 23.5°$



$$\vec{\Gamma} = \vec{R} \times (\vec{F}_+ - \vec{F}_-) = R(F_+ - F_-)(-\hat{n} \times \hat{L})$$

$$\doteq \vec{\Omega}_s \times \vec{L}$$

$\vec{\Omega}_s$ is precession of equinoxes with period

$$T_{eq} = \left(\frac{2\pi}{\Omega_s}\right) \simeq 26{,}000\ y$$

*For simplicity, replace equitorial bulge as dumbell with masses $m/2$. We can put in a correction factor later for this (dumbell → ring)

$$F_{\pm} = \frac{G M_s \, m_r/2}{(r_s \mp \cos\beta R)^2} \simeq \frac{G M_s m_r}{2 r_s^2}\left(1 \pm 2\cos\beta \frac{R}{r_s}\right)$$

$$\vec{\Gamma} = \frac{G M_s m_r R^2}{r_s^3}(2\cos\beta)(\hat{L}\times\hat{w})$$

$$|\vec{\Gamma}| \propto \cos\beta \sin\beta$$

similar factor from torque due to moon gives combined factor

$$1 + \left(\frac{r_s}{r_m}\right)^3 \frac{m_m}{m_s} \simeq 3$$

Averaging over ring rather than dumbbell model gives factor 3/8.

$$|\vec{\Gamma}| = 3\left(\frac{3}{8}\right)\frac{G M_s m_r}{r_s^3} R^2 (2\cos\beta)\sin\beta$$

$$\vec{\Gamma} = \vec{\Omega}_s \times \vec{L} = -\vec{L}\times\vec{\Omega}_s = -L\,\Omega_s\,\underbrace{(\hat{L}\times\hat{w})}_{=\sin(\beta}$$

So $\vec{\Omega}_s$ points in $-\hat{n}$ direction, opposite direction of orbital plane defined via right hand rule of earth's orbit about sun.

with $L = \lambda_3 w_e = \lambda_3 \left(\frac{2\pi}{\text{day}}\right)$

ring & sphere model $(\lambda_3 - \lambda_1) = \frac{1}{2} m_r R^2$

So $m_r R^2 = 2(\lambda_3 - \lambda_1)$

For circular earth orbit,

$$r_s \omega_0^2 = \frac{G m_s}{r_s^2} \qquad \omega_0 = \frac{2\pi}{y}$$

$$\Omega_s = \left(\frac{1}{\lambda_3 \omega_e}\right)\left(\frac{18}{8}\right)\underbrace{\boxed{\frac{G m_s}{r_s^3}}}_{\omega_0^2}\ \underbrace{\boxed{m_r R^2}}_{2(\lambda_3 - \lambda_1)} \cos\beta$$

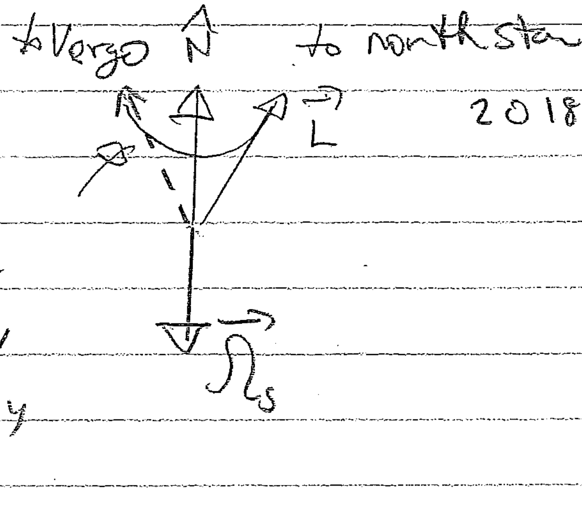
$$= \frac{18}{4}\underbrace{\left(1 - \frac{\lambda_1}{\lambda_3}\right)}_{1/306}\frac{\omega_0^2}{\omega_e}\cos\beta \qquad \beta = 23.5^\circ$$

$$\left(\frac{\omega_0}{\omega_e}\right) = \frac{1}{365}$$

$$\frac{2\pi}{T_{eq}} = \Omega_s = \frac{18}{4}\left(\frac{1}{306}\right)\left(\frac{1}{365}\right)\underbrace{\cos(23.5^\circ)}_{0.92}\left(\frac{2\pi}{y}\right)$$

$$= \frac{1}{27{,}000}\left(\frac{2\pi}{y}\right)$$

$T_{eq} = 27{,}000\ y$. Very close to observed $26{,}000\ y$



13,000 y
+ 2018 y
= 15,018 y