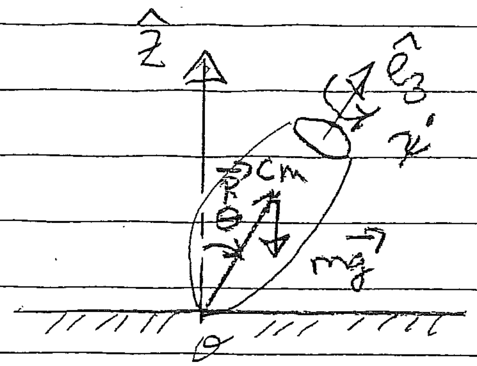
Physics 304 Spring 2018

Lecture #12 : Top with Torque



Point of conact, origin O is fixed.

Height of C.M. $= R\cos\theta$

$U(\theta) = mgR\cos\theta$

$\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{inertial} \perp \hat{z} \Rightarrow \frac{dL_z}{dt} = 0$

Gravity acts at cm (uniform field), no torque along body axis $\hat{e}_3$. Ignore friction at point of contact, so $\dot{\psi}$ = constant.

Write Lagrangian. Top has 3 degrees of freedom, Euler angles (inertial frame coordinates). Our definition:

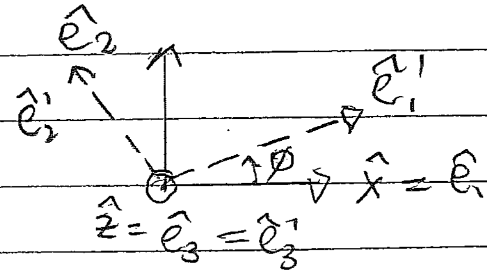
1) rotate body about $\hat{z}$ by ϕ until $\hat{e}_2'$ points in direction of θ rotation

2) rotate about $\hat{e}_2'$ ("line of nodes") by θ

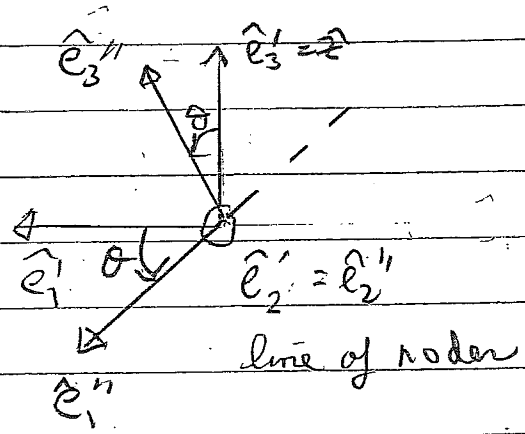
3) rotate about body symmetry axis by ψ

All rotations are right handed rotations.

matricies rotate coordinates.

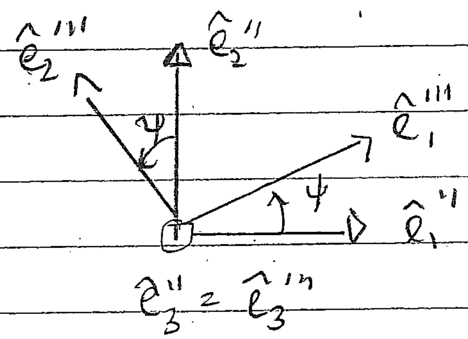
(1) 

$$R_1 = \begin{pmatrix} c_\phi & s_\phi & 0 \\ -s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) 

$$R_2 = \begin{pmatrix} c_\theta & 0 & -s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{pmatrix}$$

watch minus sign!

(3) 

$$R_3 = \begin{pmatrix} c_\psi & s_\psi & 0 \\ -s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix}_{\text{body}}^{'''} = R_3(\psi)\, R_2(\theta)\, R_1(\phi) \begin{pmatrix} \\ \\ \\ \end{pmatrix}_{\substack{\text{inertia} \\ \text{or space}}}$$

Example Write body axis $\hat{e}_3''' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{body}$

in inertial basis $\hat{x}, \hat{y}, \hat{z}$

$$\begin{pmatrix} \\ \\ \end{pmatrix}_{inertial} = R_1^{-1} R_2^{-1} R_3^{-1} \begin{pmatrix} \\ \\ \end{pmatrix}_{body}$$

R_3^{-1} does nothing to $\hat{e}_3'''$

$$\begin{pmatrix} \\ \\ \end{pmatrix}_{inertial} = R_1^{-1} R_2^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = R_1^{-1} \begin{pmatrix} S\theta \\ 0 \\ C\theta \end{pmatrix}$$

$$= \begin{pmatrix} C_\psi S\theta \\ S_\psi C\theta \\ C_\theta \end{pmatrix}$$

$$\hat{e}_3''' = (\hat{x} C_\psi + \hat{y} S_\psi) S_\theta + \hat{z} C_\theta$$

usual polar coordinates

$$\mathcal{L} = \frac{1}{2}\vec{\omega}\cdot\bar{I}\cdot\vec{\omega} - mgR\cos\theta$$

In body frame, $\bar{I} = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_3)$ (for top)

$\vec{\omega}$ is rotation about respective axes:

$$\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3''$$

$$= \omega_1\hat{e}_1''' + \omega_2\hat{e}_2''' + \omega_3\hat{e}_3'''$$

$\hat{z}$: $R_2(\theta)R_1(\phi)\begin{pmatrix}0\\0\\1\end{pmatrix} = R_2\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-S_\theta\\0\\C_\theta\end{pmatrix}$

$$\hat{z} = -S_\theta\hat{e}_1'' + C_\theta\hat{e}_3'' \quad *$$

$$\vec{\omega} = -S_\theta\dot{\phi}\,\hat{e}_1'' + \dot{\theta}\hat{e}_2'' + (\dot{\phi}C_\theta + \dot{\psi})\,\hat{e}_3''$$

Now last rotation doesn't change $\hat{e}_3''$. For top,

$$\frac{1}{2}\vec{\omega}\cdot\bar{I}\cdot\vec{\omega} = \frac{1}{2}\lambda_1(\omega_1^2+\omega_2^2) + \frac{1}{2}\lambda_3\omega_3^2$$

$\omega_1^2+\omega_2^2$ is invariant under last rotation about $\hat{e}_3''$, so we need not do it. We have what we want - <u>no need to do $R_3(\psi)$ rotation</u>.

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* This is Taylor 10.98 written $\hat{z} = -S_\theta\hat{e}_1' + C_\theta\hat{e}_3$ the way he uses primes

Lagrangian in inertial coordinates (Euler angles)

$$\mathcal{L} = \frac{1}{2}\lambda_1\left(S_\theta^2\dot\phi^2 + \dot\theta^2\right) + \frac{1}{2}\lambda_3\left(\dot\phi C_\theta + \dot\psi\right)^2$$

$$-mgR\cos\theta$$

We see ϕ, ψ are cyclic.

$\psi\rfloor$ $$P_\psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = \lambda_3\left(\dot\phi C_\theta + \dot\psi\right) = \text{Constant} \equiv L_3$$

$\phi\rfloor$ $$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \lambda_1 S_\theta^2\dot\phi + \lambda_3 C_\theta\left(\dot\phi C_\theta + \dot\psi\right) \equiv L_z$$

Note P_ψ is component of $\vec{L}$ along body symmetry axis $\hat{e}_3$. (We now drop primes, $\hat{e}_i$ are final body axes)

$$L_3 = \lambda_3 W_3 = \lambda_3\left(\dot\phi C_\theta + \dot\psi\right) = P_\psi$$

$\theta\rfloor$ $$\lambda_1\ddot\theta = \lambda_1 S_\theta C_\theta\dot\phi^2 + \underbrace{\lambda_3\left(\dot\phi C_\theta + \dot\psi\right)}_{P_\psi = L_3 \text{ constant}}(-S_\theta)\dot\phi + mgRS_\theta$$

$\phi\rfloor$ equation,

$$\dot\phi = \frac{P_\phi - C_\theta P_\psi}{\lambda_1 S_\theta^2} \equiv \Omega(\theta)$$

So $\lambda_1\ddot\theta = f(\theta)$ albeit complicated f

General solution both precession (rotation about $\hat{z}$) and nutation (oscillation about $\hat{e}_1'$ line of nodes).
Energy equation is simpler to analyize.
(take $s = s_\theta = \sin\theta$ and $c = c_\theta = \sin\theta$)

$$E = \frac{1}{2}\lambda_1\left(s^2\dot{\phi}^2 + \dot{\theta}^2\right) + \frac{1}{2}\frac{P_\psi^2}{\lambda_3} + mgRC$$

$$E - \frac{1}{2}\frac{P_\psi^2}{\lambda_3} \equiv E' \quad \text{constant}$$

$\frac{1}{2}\lambda_1\dot{\theta}^2$ is θ kinetic energy term

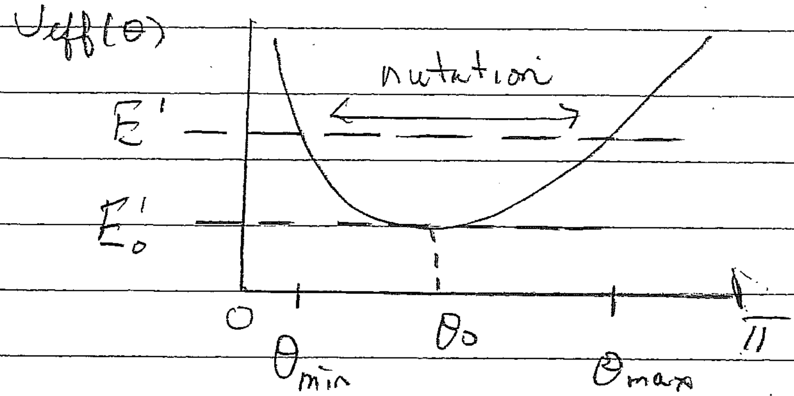
$$\dot{\phi} = \frac{P_\phi - C P_\psi}{\lambda_1 s^2} \equiv \Omega(\theta) \quad \text{function of } \theta$$

$$E' = \frac{1}{2}\lambda_1\dot{\theta}^2 + \underbrace{\frac{1}{2}\lambda_1 s^2\dot{\phi}^2 + mgRC}_{\equiv\, U_{eff}(\theta)}$$

$\equiv U_{eff}(\theta)$ effective potential energy

$$E' = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{eff}(\theta)$$

sketch of $U_{eff}(\theta)$



θ_0 is solution we found earlier, $\dot{\theta} = 0$

Solution $\theta = \theta_0$ constant (exists if quadratic in $\dot\phi$ has real roots). Return to Lagrange equation of motion. $(S_0 = \sin\theta_0,\ C_0 = \cos\theta_0)$

$$\lambda_1 \ddot\theta = 0 = \lambda_1 S_0 C_0 \dot\phi^2 - P_\psi S_0 \dot\phi + mgR S_0$$

remember that $\dot\phi = \dfrac{P_\phi - C_0 P_\psi}{\lambda_1 S_0^2} = \Omega(\theta_0)$ function of θ_0

Solve quadratic for $\dot\phi$ —

$$\dot\phi^2 - \frac{P_\psi}{\lambda_1 C_0}\dot\phi + \frac{mgR}{C_0\lambda_1} = 0$$

$$\dot\phi_\pm = \frac{1}{2}\left(\frac{P_\psi}{\lambda_1 C_0}\right) \pm \frac{1}{2}\left[\left(\frac{P_\psi}{\lambda_1 C_0}\right)^2 - \frac{4mgR}{C_0\lambda_1}\right]^{1/2}$$

$$= \frac{1}{2}\left(\frac{P_\psi}{\lambda_1 C_0}\right)\left[1 \pm \sqrt{1 - a^2}\right]^{1/2}$$

$$a \equiv \frac{4mgR\lambda_1 C_0}{P_\psi^2}$$

take $a \ll 1$ $\quad \dot\phi_\pm \cong \frac{1}{2}\left(\frac{P_\psi}{\lambda_1 C_0}\right)\left[1 \pm 1 \mp \frac{1}{2}a^2\right]$

$$= \begin{cases} P_\psi/\lambda_1 C_0 \\ \dfrac{P_\psi}{\lambda_1 C_0}\dfrac{1}{4}a^2 = \dfrac{mgR}{P_\psi} \end{cases}$$

slow precession $\dot\phi_- = \dfrac{mgR}{P_\psi}$

fast precession $\dot\phi_+ = \dfrac{P_\psi}{\lambda_1 \cos(\theta_0)}$

From Barger & Olsson Classical Mechanics

precession frequency $\omega_p \equiv \dot{\phi}_- = \frac{mgR}{p_\psi}$

precession at $t=0$ $\omega_0 \equiv \dot{\phi}(0)$

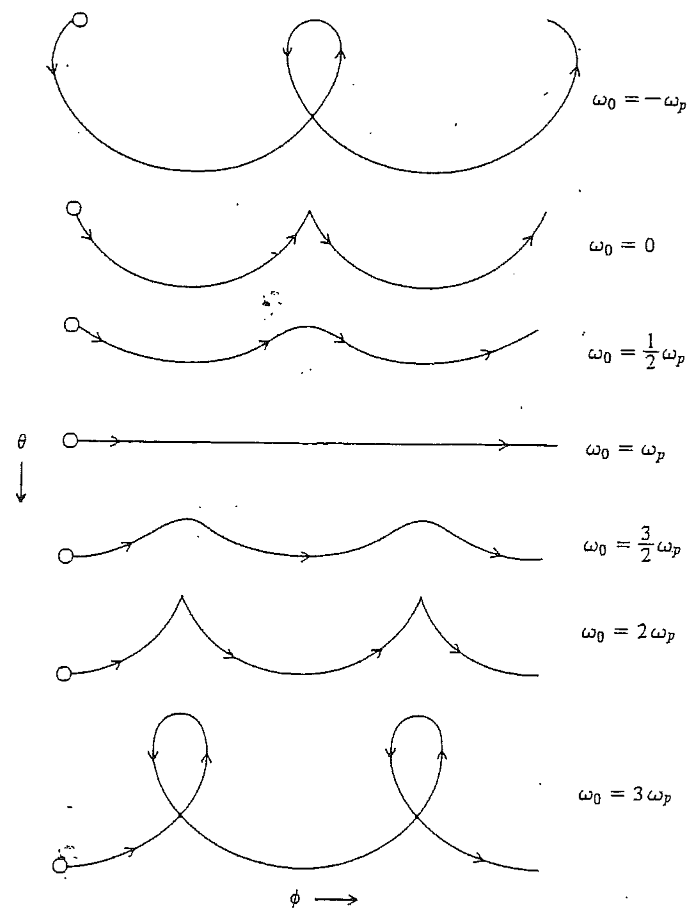


FIGURE 7-22. Nutation curves traced out by the symmetry axis of the top for various initial conditions. The top is started at the same value of θ in each case and the resulting curves are trochoids.