Physics 304 Spring 2019

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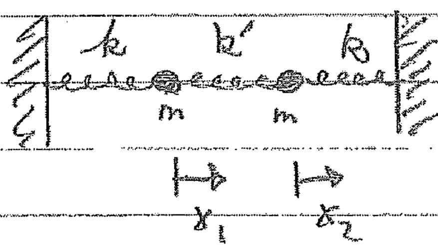
Lec13: Coupled Oscillations I

Very important problem. similar math as diagonalizing moment of inertia tensor

simple example:



generalised coordinates are displacements from equilibrium

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k'(x_1-x_2)^2$$

note, unstretched lengths do not appear when coordinates are displacements from equilibrium

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$m\ddot{x}_1 = -kx_1 - k'(x_1-x_2) = -(k+k')x_1 + k'x_2$$

$$m\ddot{x}_2 = -kx_1 + k'(x_1-x_2) = -(k+k')x_2 + k'x_1$$

Eventually, we will consider many linearly coupled oscillators, so convenient to use matricies.

$$\frac{d^2}{dt^2}\left[\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] = -\begin{pmatrix} k+k', & -k' \\ -k', & k+k' \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\overline{m}\cdot\ddot{\vec{x}} = -\overline{k}\cdot\vec{x}$

look for solutions with single frequency.

let $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t}$ a_1, a_2 complex constants

$$\begin{bmatrix} -m\omega^2 + (k+k'), & -k' \\ -k', & -m\omega^2 + (k+k') \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

det[ ] = 0 and solve quadratic

$$\left[m\omega^2 - (k+k')^2\right]^2 = k'^2$$

$$m\omega_\pm^2 - (k+k')^2 = \pm k'$$

$$\omega_\pm^2 = \frac{(k+k') \pm k'}{m} = \begin{cases} \frac{k+2k'}{m} \\ \frac{k}{m} \end{cases}$$

low and high frequency solutions. normalize as

$$\omega_-: \quad a_1 = a_2 = \frac{1}{\sqrt{2}}, \quad \vec{a}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_+: \quad a_1 = -a_2 = \frac{1}{\sqrt{2}}, \quad \vec{a}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\vec{a}_i$ are normal modes (eigenvectors)

$$\vec{X}(t) = \underset{\text{real part}}{R_e} \sum_{i=\pm} \xi_i(t)\,\vec{a}_i = R_e\left\{ \xi_+(t)\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} + \xi_-(t)\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}$$

expansion coefficients $\xi_i(t)$ are called <u>normal coordinates</u>

$$\xi_i(t) = Z_i e^{i\omega_i t} = c_i e^{i(\omega_i t - \delta_i)} \qquad c_i, \delta_i \text{ real}$$

$$R_e(\xi_i) = c_i \cos(\omega_i t - \delta_i)$$

<u>Orthogonality</u>: $\vec{a}_i \cdot \vec{a}_j = \vec{a}_i^{\,\underset{\text{transpose}}{T}} \cdot \vec{a}_j = \delta_{ij}$

<u>More generally</u> for $\bar{m}$ not proportional to identity

$$\vec{a}_i \cdot \bar{m} \cdot \vec{a}_j = \delta_{ij}$$

<u>Specific solution given initial conditions</u>

take $\vec{X}(0) = \begin{pmatrix}a\\b\end{pmatrix}$ $\qquad \dot{\vec{X}}(0) = 0$

$$\vec{X}(0) = \sum_i c_i \cos\delta_i \,\vec{a}_i$$

$$\dot{X}(0) = \sum_i \omega_i c_i \sin\delta_i \,\vec{a}_i$$

$$\vec{a}_i \cdot \vec{X}(0) = c_i \cos\delta_i$$

$$\vec{a}_i \cdot \dot{\vec{X}} = \omega_i c_i \sin\delta_i$$

with $\dot{\vec{X}}(0) = 0$, $\quad \delta_i = 0$

$$\vec{a}_\pm \cdot \vec{x}(0) = \frac{1}{\sqrt{2}}(1, \mp 1)\begin{pmatrix} a \\ b \end{pmatrix} = c_\pm$$

$$\Rightarrow \quad c_\pm = \frac{1}{\sqrt{2}}(a \mp b)$$

$$\vec{x}(t) = \frac{1}{2}(a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\omega_+ t + \frac{1}{2}(a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_- t$$

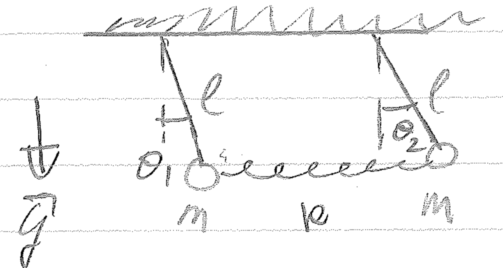
$$\boxed{\begin{aligned} x_1(t) &= \frac{1}{2}(a-b)\cos\omega_+ t + \frac{1}{2}(a+b)\cos\omega_- t \\ x_2(t) &= -\frac{1}{2}(a-b)\cos\omega_+ t + \frac{1}{2}(a+b)\cos\omega_- t \end{aligned}}$$

<u>note</u> $a=b$ excites ω_- mode (even)
$a=-b$ excites ω_+ mode (odd)

Space of solutions is a <u>linear vector space</u> with <u>normal modes</u> as orthonormal basis and <u>normal coordinates</u> as vector coefficients.

Easily generalizes to arbitrary number of dimensions for $\vec{x}$.

Example: Coupled pendulums



massless spring k
massless rods l

$$T = \tfrac{1}{2} m l^2 \left(\dot\theta_1^2 + \dot\theta_2^2\right)$$

$$U = mgl\left(1-\cos\theta_1\right) + mgl\left(1-\cos\theta_2\right) + \tfrac{1}{2} k l^2 \left(\sin\theta_1 - \sin\theta_2\right)$$

$$\approx \frac{mgl}{2}\left(\theta_1^2+\theta_2^2\right) + \tfrac{1}{2} k l^2 \left(\theta_1-\theta_2\right)^2$$

small angles

Equations of motion $\quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot\theta_i} = \dfrac{\partial L}{\partial \theta_i}$

$$m l^2 \ddot\theta_1 = -mgl\,\theta_1 - k l^2(\theta_1-\theta_2)$$

$$m l^2 \ddot\theta_2 = -mgl\,\theta_2 + k l^2(\theta_1-\theta_2)$$

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = -\begin{bmatrix} g/l + \frac{k}{m}, & -\frac{k}{m} \\ -k/m, & \frac{g}{l}+\frac{k}{m}\end{bmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

det[ ] = 0 gives $\left[\omega^2 - \left(\frac{g}{l}+\frac{k}{m}\right)\right]^2 - \frac{k^2}{m^2} = 0$

$$\omega_+^2 = \frac{g}{\ell} + \frac{2k}{m} \qquad \vec{a}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \text{ anti-symmetric}$$

$$\omega_-^2 = \frac{g}{\ell} \qquad \vec{a}_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \text{ symmetric}$$

<u>Weak coupling (beats)</u>

$\frac{k}{m} \ll \frac{g}{\ell}$ define $\omega_0^2 \equiv \frac{g}{\ell} + \frac{k}{m}$

natural frequency of oscillator with other held fixed.

let $\frac{1}{2}\left(\frac{k}{m}\right)/\left(\frac{g}{\ell}\right) \equiv \varepsilon$

then $\omega_0 = \sqrt{\frac{g}{\ell}}(1+2\varepsilon)^{1/2} \doteq \sqrt{\frac{g}{\ell}}(1+\varepsilon)$

solutions $\omega_+^2 = \frac{g}{\ell} + 2\frac{k}{m} = \frac{g}{\ell}(1+4\varepsilon)$

$$\omega_+ = \sqrt{\frac{g}{\ell}}(1+2\varepsilon) \cong \omega_0(1+\varepsilon)$$

drop ε^2

and $\omega_- = \sqrt{\frac{g}{\ell}} = \omega_0(1-\varepsilon)$

so ω_+, ω_- split by $2\omega_0\varepsilon$

$$\omega_0 \;\text{---}\!\!< \begin{array}{l} \omega_+ \\ \updownarrow 2\omega_0\varepsilon \\ \omega_- \end{array}$$

<u>Take</u> initial condition

$$\vec{\theta}(0) = \begin{pmatrix}\theta_0\\0\end{pmatrix}; \quad \dot{\vec{\theta}}(0) = \begin{pmatrix}0\\0\end{pmatrix}$$

Complex $q(t) = \frac{\theta_0}{\sqrt{2}} e^{i\omega_+ t} \vec{a}_+ + \frac{\theta_0}{\sqrt{2}} e^{i\omega_- t} \vec{a}_-$

$$= \frac{\theta_0}{\sqrt{2}} e^{i\omega_0 t} \left[e^{i\varepsilon\omega_0 t} \vec{a}_+ + e^{-i\varepsilon\omega_0 t} \vec{a}_- \right]$$

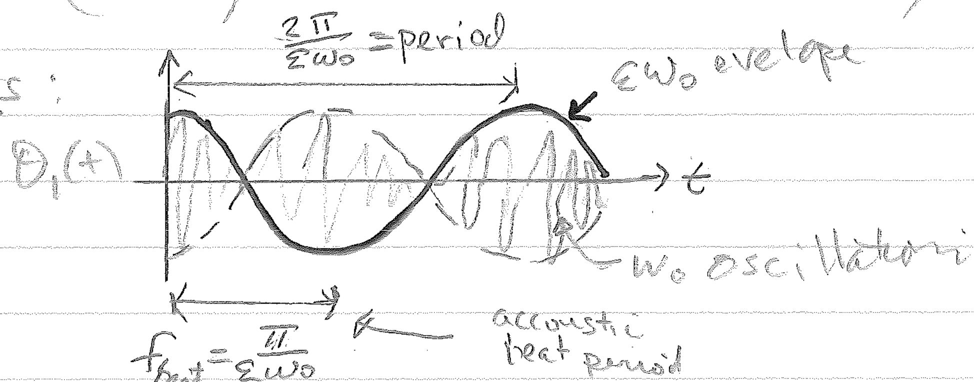
$$= \frac{\theta_0}{2} e^{i\omega_0 t} \begin{pmatrix} e^{i\varepsilon\omega_0 t} + e^{-i\varepsilon\omega_0 t} \\ -e^{i\varepsilon\omega_0 t} + e^{-i\varepsilon\omega_0 t} \end{pmatrix}$$

$$= \theta_0 e^{i\omega_0 t} \begin{pmatrix} \cos(\varepsilon\omega_0 t) \\ -i\sin(\varepsilon\omega_0 t) \end{pmatrix}$$

take real part

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \theta_0 \begin{pmatrix} \cos\omega_0 t \cos\varepsilon\omega_0 t \\ \sin\omega_0 t \sin\varepsilon\omega_0 t \end{pmatrix}$$

__beats__:



__Note.__ for accoustic beats you hear __intensity modulation__ at the beat frequency

$$\text{accoustic } f_{beat} = \frac{\pi}{\varepsilon\omega_0} = \left(\frac{2\pi}{\omega_+ - \omega_-} \right)$$

General linear problem T, U quadratic form

$$\mathcal{L} = \frac{1}{2}\sum_{i,j} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i,j} k_{ij} q_i q_j$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i,j}\left(m_{ij}\dot{q}_i\delta_{jk} + m_{ij}\delta_{ik}\dot{q}_j \right)$$

$$= \sum_i m_{ki}\dot{q}_i \quad \text{for } m_{ij} = m_{ji}$$

similarly for $k_{ij} = k_{ji}$

$$\frac{\partial \mathcal{L}}{\partial q_k} = -\sum_i k_{ki} q_i$$

k^{th} equation of motion,

$$\sum_i m_{ki}\ddot{q}_i = -\sum_i k_{kj} q_j$$

or in matrix form

$$\bar{m}\cdot\ddot{\vec{q}} = -\bar{k}\cdot\vec{q}$$

$\bar{m}$ "mass"

$\bar{k}$ "spring constant"

as before with $\vec{q}(t) = \mathrm{Re}\{\vec{a}\, e^{i\omega t}\}$

$$\left(-\omega^2\bar{m} + \bar{k}\right)\cdot\vec{a} = 0$$

$$\det\underbrace{\left(-\omega^2\bar{m} + \bar{k}\right)}_{N\times N \text{ matrix}} = 0$$

n^{th} order polynomial

Solution

$$\vec{q}(t) = \sum_i C_i \cos(\omega_i t - \delta_i)\, \vec{a}_i$$