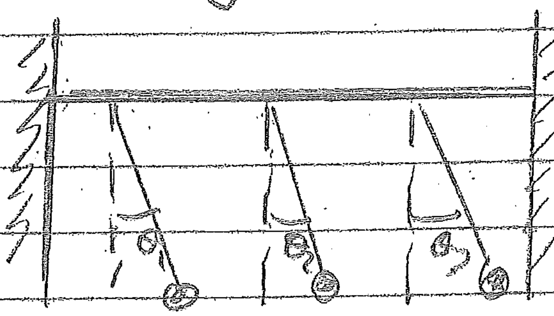
Lecture 14: Coupled Oscillators II

3 coupled pendulums - an example of degeneracy.



flexible support provides coupling

$$T = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$$

$$U = \frac{1}{2}\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right) + \frac{1}{2}\varepsilon(\theta_1 - \theta_2)^2$$

$$+ \frac{1}{2}\varepsilon(\theta_1 - \theta_3)^2 + \frac{1}{2}\varepsilon(\theta_2 - \theta_3)^2$$

$$\simeq \frac{1}{2}\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right) - \varepsilon\left(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3\right)$$

$$\underline{\underline{m}} = \underline{\underline{1}}, \quad \underline{\underline{k}} = \begin{bmatrix} 1 & -\varepsilon & -\varepsilon \\ -\varepsilon & 1 & -\varepsilon \\ -\varepsilon & -\varepsilon & 1 \end{bmatrix}$$

$$-\omega^2 \underline{\underline{m}} + \underline{\underline{k}} =$$

$$\begin{pmatrix} 1-\omega^2 & -\varepsilon & -\varepsilon \\ -\varepsilon & 1-\omega^2 & -\varepsilon \\ -\varepsilon & -\varepsilon & 1-\varepsilon^2 \end{pmatrix}$$

det[ ] = 0 gives

$$(1-\omega^2)\left[(1-\omega^2)^2-\varepsilon^2\right]$$

$$+\varepsilon\left[-\varepsilon(1-\varepsilon^2)-\varepsilon^2\right]$$

$$-\varepsilon\left[\varepsilon^2+\varepsilon(1-\omega^2)\right]=0$$

$$(1-\omega^2)^3-2\varepsilon^3-3\varepsilon^2(1-\omega^2)=0$$

$$(\omega^2-1-\varepsilon)^2(\omega^2-1+2\varepsilon)=0$$

3 roots are:

$$\left.\begin{array}{l}\omega_1=\sqrt{1+\varepsilon}\\ \omega_2=\sqrt{1+\varepsilon}\end{array}\right\}\text{degeneracy}$$

$$\omega_3=\sqrt{1-2\varepsilon}$$

Eigen vectors: $\vec{a}_3=\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}$

row index

$$\begin{pmatrix}2\varepsilon & -\varepsilon & -\varepsilon\\ -\varepsilon & 2\varepsilon & -\varepsilon\\ -\varepsilon & -\varepsilon & 2\varepsilon\end{pmatrix}\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}=0$$

$$2a_{13}-a_{23}-a_{33}=0$$

$$-a_{13}+2a_{23}-a_{33}=0$$

subtract equations to eliminate a_{33}

$$3a_{13} - 3a_{23} = 0 \quad \Rightarrow a_{13} = a_{23}$$

and then $\quad a_{33} = a_{13} = a_{23}$

$$\vec{a_3} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\vec{a_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$ and so

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = 0$$

$$a_{11} + a_{21} + a_{31} = 0$$

similarly, for $\vec{a_2}$

$$a_{12} + a_{22} + a_{32} = 0$$

Because of degeneracy, these eigenvectors are not uniquely determined. If we choose $a_{31} = 0$

$$\vec{a_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{a_2} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

where we used orthogonality for $\bar{m} = \bar{1}$ (unit matrix)

$$\vec{a_i} \cdot \vec{a_j} = \delta_{ij}$$

Orthogonality, general case (see Appendix)

$$(\vec{a}_j)_i = a_{ij}$$

for simplicity, set $C_i = 1$, $\delta_i = 0$ so

$$\vec{q}_i = e^{i\omega_i t}\vec{a}_i$$

$$\overline{m}\cdot\ddot{\vec{q}} = -\overline{k}\cdot\vec{q}_i \quad \text{gives}$$

$$\omega_i^2\,\overline{m}\cdot\vec{a}_i = +\overline{k}\cdot\vec{a}_i$$

and $$\omega_j^2\,\overline{m}\cdot\vec{a}_j = +\overline{k}\cdot\vec{a}_j$$

with symmetry of $\overline{m}$, $\overline{k}$ get

$$(\omega_i^2 - \omega_j^2)\,\vec{a}_i\cdot\vec{m}\cdot\vec{a}_j = 0$$

for $i \neq j$ $\quad \vec{a}_i\cdot\overline{m}\cdot\vec{a}_j = 0$

for $i = j$, since kinetic energy is positive definite

$$\vec{a}_i\cdot\overline{m}\cdot\vec{a}_i > 0$$

thus $\quad \vec{a}_i\cdot\overline{m}\cdot\vec{a}_j = \delta_{ij}$

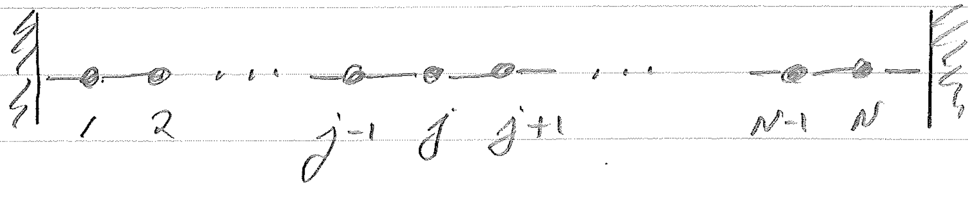
Since in $\vec{a}_i$ basis, $\bar{m}$ is diagonal, we could always define

$$\vec{a}_i{}' = \frac{1}{\sqrt{m_{ii}}}\vec{a}_i$$

and have $\vec{a}_i{}' \cdot \vec{a}_j{}' = \delta_{ij}$

Loaded string. Newton (1687); Bernoulli + son (1727)

N masses (m) connected by $N+1$ elastic (tension τ) strings between fixed ends

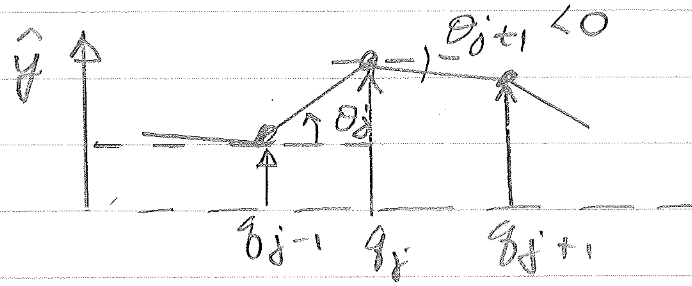


spacing d and total length $L=(N+1)d$

Let vertical displacements be q_j

$\sin\theta_j \simeq \tan\theta_j$

$\theta_j \cong \dfrac{q_j - q_{j-1}}{d}$



For small displacements, restoring tension in $\hat{y}$ direction

$$F_{y_j} = -\tau \sin\theta_j + \tau \sin\theta_{j+1}$$

$$= -\frac{\tau}{d}(q_j - q_{j-1}) + \frac{\tau}{d}(q_{j+1} - q_j)$$

$$= -\frac{\tau}{d}(2q_j - q_{j-1} - q_{j+1})$$

j^{th} equation of motion

$$\ddot{q}_j = -\frac{\tau}{md}(2q_j - q_{j-1} - q_{j+1}) \qquad q_0 \equiv 0$$

$$q_{N+1} \equiv 0$$

or potential $U = \frac{1}{2}\frac{\tau}{d}\sum_{j=1}^{N+1}(q_{j-1} - q_j)^2$

Nearest neighbor coupling

$$\underline{K} = \frac{\tau}{d}\begin{bmatrix} 2, -1, 0, 0 & & \\ -1 & 2 -1, & 0, \ldots \\ 0 & -1 & 2 -1, \ldots \\ 0 & \ldots & \\ \vdots & & \end{bmatrix}_{n \times n}$$

$$\det\left[\underline{m}\omega^2 - \underline{K}\right] = 0$$

$\underline{N=1}$ $\quad m\omega^2 - 2\tau/d = 0 \quad ; \omega^2 = \left(\frac{2\tau}{md}\right)$

$\underline{n=2}$

$$\det\begin{pmatrix} \omega^2 - 2\tau/md & , +\tau/md \\ +\tau/md & , \omega^2 - 2\tau/md \end{pmatrix} = 0$$

$$\omega^2 = \frac{2\tau}{md} \pm \frac{\tau}{md}$$

general case (see manuscript)

try $\quad q_j = a\, e^{i(j\gamma - \delta)}\, e^{i\omega t} \quad a, \gamma, \delta$ real

j^{th} equation of motion

$$\ddot{q}_j = \left(\frac{\tau}{md}\right)\left(q_{j-1} - 2q_j + q_{j+1}\right)$$

substitution gives (cancel $a e^{i(j\gamma - \delta)}$)

$$-\omega^2 = \left(\frac{\tau}{md}\right)\left[e^{-i\gamma} - 2 + e^{i\gamma}\right]$$

$$\omega^2 = 2\left(\frac{\tau}{md}\right)(1-\cos\gamma)$$

$$\omega^2 = 4\left(\frac{\tau}{md}\right)\sin^2\frac{\gamma}{2}$$

$$\omega = 2\sqrt{\frac{\tau}{md}}\sin\left(\frac{\gamma}{2}\right)$$

there will be n of these, one for each root of determinant (normal mode)

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left(\frac{\gamma_r}{2}\right) \qquad r=1,\ldots,n$$

with corresponding normal mode $\vec{a}_r$, an N component column vector.

Solutions are of the form:

$$(\vec{a}_r)_j \equiv a_{jr} = a_r e^{i(j\gamma_r - \delta_r)}$$

real part $\mathrm{Re}(\vec{a}_r)_j = a_r\cos(j\gamma_r - \delta_r)$

with $a_{0r} = a_{(n+1)r} = 0$

boundary conditions

a_{0r} determines $\delta_r = \pi/2$.

$$(\vec{a}_r)_j = a_r\sin(j\gamma_r)$$

$(\vec{q}_r)_{N+1} \Rightarrow$ means $q_{(n+1),r} = 0$

γ_r discrete ("quantized")

$$a_{(N+1)r} = a_r \sin[(N+1)\gamma_r] = 0$$

$$(N+1)\gamma_r = s\pi \qquad s = 1, 2, \ldots, N \text{ normal modes}$$

1-1 correspondence, so define them equal

$$\boxed{\gamma_r = \left(\frac{r\pi}{N+1}\right)}$$

can choose as 1 since multiplier C_r

$$(\vec{a}_r)_j = a_{jr} = a_r \sin\left(\frac{jr\pi}{N+1}\right)$$

$$\omega_r = 2\sqrt{\frac{\tau}{md}} \sin\left(\frac{\gamma_r}{2}\right) = 2\sqrt{\frac{\tau}{md}} \sin\left(\frac{r\pi}{2(N+1)}\right)$$

$$= 2\sqrt{\frac{\tau}{md}} \sin\left(\frac{r\pi}{2}\frac{d}{L}\right)$$

general solution is $\underline{\text{linear}}$ superposition **

$$(\vec{q})_j = \sum_r C_r \cos(\omega_r t - \delta_r)(\vec{a}_r)_j$$

where position of j^{th} mass is $x_j = jd$

$$q(x_j, t) = \sum_r C_r \cos(\omega_r t - \delta_r) \sin\left(\frac{r\pi x_j}{L}\right)$$

** note that this delta is initial condition in time, not phase we previously set to pi/2!!

Orthogonality in this case $\overline{M} = m\,\overline{I}$ so

$$\vec{a}_r \cdot \vec{a}_s = \sum_j a_{rj}\, a_{js} = (\text{const})\,\delta_{rs}$$

transpose

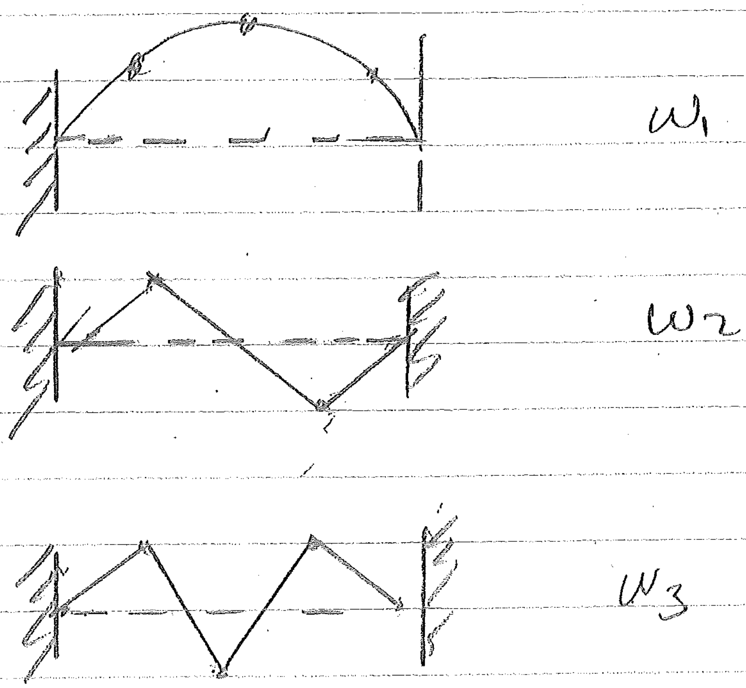
$$\sum_j \sin\left[j\left(\frac{r\pi}{N+1}\right)\right] \sin\left[j\left(\frac{s\pi}{N+1}\right)\right] = \frac{N+1}{2}\,\delta_{rs}$$

So, just as before, get C_r, δ_r from

$$\sum_j q_j(0) \sin\left[j\left(\frac{r\pi}{N+1}\right)\right] = \left(\frac{N+1}{2}\right) C_r \cos(\delta_r)$$

$$\sum_j \dot{q}_j(0) \sin\left[j\left(\frac{r\pi}{N+1}\right)\right] = \left(\frac{N+1}{2}\right) C_r\, \omega_r \sin(\delta_r)$$

modes for $N=3$:



note resemblance to standing waves on string.

Continuous limit: $N \to \infty$, $d \to 0$ s.t. $(N+1)d = L$

$m \to 0$, $d \to 0$ s.t. $\frac{m}{d} = \lambda$ $\left(\frac{\text{mass}}{\text{length}}\right)$

$$a_{jr} = \sin\left(\frac{jr\pi}{N+1}\right) = \sin\left[r\pi\left(\frac{jd}{(N+1)d}\right)\right]$$

with x=jd

$$\longrightarrow \sin\left(\frac{r\pi x}{L}\right)$$ normal modes, or eigenvectors!

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left(\frac{r\pi}{2(N+1)}\right) = \sqrt{\frac{\tau}{\lambda}}\frac{2}{d}\sin\left(\frac{r\pi d}{2L}\right)$$

$$\longrightarrow \sqrt{\frac{\tau}{\lambda}}\left(\frac{r\pi}{L}\right)$$ still discrete, due to boundary conditions

boundary condition are fixed ends

$$q(0,t)=0 \ , \quad \dot{q}(0,t)=0$$
$$q(L,t)=0 \ , \quad \dot{q}(L,t)=0$$

orthogonality of eigenvectors (eigenfunctions)

$$\int_0^L dx \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right) = \frac{L}{2}\,\delta_{rs}$$

general solution is superposition of normal modes due to linearity of problem

$$q(x,t) = \sum_{r=1}^{\infty} C_r \cos(\omega_r t - \delta_r)\,\sin\left(\frac{r\pi x}{L}\right)$$

$$\omega_r = \sqrt{\frac{\tau}{\lambda}}\left(\frac{r\pi}{L}\right)$$

discrete frequencies, but no limit on value (r)

[but, eventually before $r \to \infty$, string will break!]