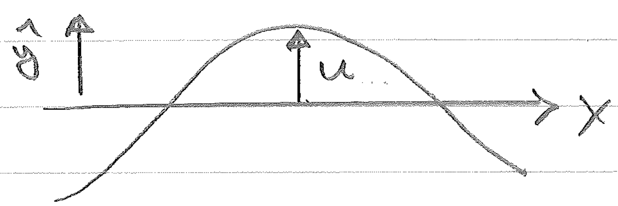
Physics 304

Lecture #15: Wave Equation

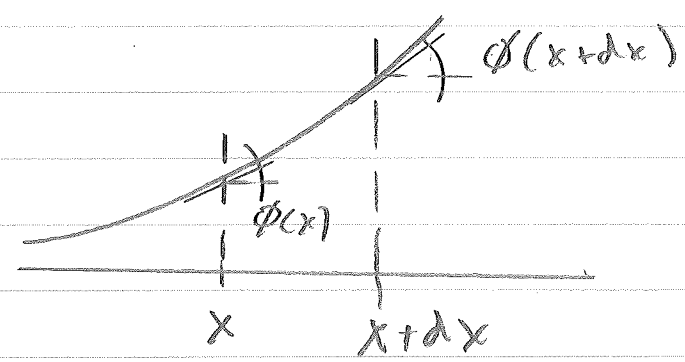
start of Taylor ch 16

Transverse waves on taut string – tension T, mass/length λ, displacement from equilibrium ("wave amplitude") $u(x,t)$



$y=0$ is equilibrium

Force on small section



$$F_y = T\left[\sin(\phi(x+dx)) - \sin(\phi(x))\right]$$

$\sin\phi \approx \tan\phi \approx \phi = \dfrac{\partial u}{\partial x} \equiv u'$ for small amplitudes u

mass of this string section is λdx

$$\lambda dx \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial u}{\partial x}\Big|_{x+dx} - \frac{\partial u}{\partial x}\Big|_{x}\right] = T dx\, \phi'$$

expand $\phi(x) = \dfrac{\partial u}{\partial x}$

$$\phi(x+dx) \cong \phi(x) + dx\, \phi'(x)$$

$$\lambda \frac{\partial^2 u}{\partial t^2} = T\phi'(x) = T\frac{\partial^2 u}{\partial x^2}$$

Convenient to write $\frac{\partial u}{\partial t} = \dot{u}$ and $\frac{\partial u}{\partial x} = u'$

$$\ddot{u} = \left(\frac{T}{\lambda}\right) u''$$

$T/\lambda \equiv c^2$ has dimension of $(\text{speed})^2$ and is speed of wave propagation.

Solution by separation of variables (p16.12)

let $u(x,t) = g(x)f(t)$

gives $\underbrace{\frac{\ddot{f}}{f}}_{\text{all } t} = c^2 \underbrace{\frac{g''}{g}}_{\text{all } x} = \text{constant}$

$f(t) = e^{\pm i\omega t}$ $\qquad \omega = 2\pi(\text{frequency}) = \frac{2\pi}{(\text{period})}$

$g(x) = e^{\pm ikx}$ $\qquad k = \omega/c$ [inverse length]
wave number

solutions are real parts of linear combinations

$$\exp[\pm ikx \pm i\omega t] \quad \text{all 4 combinations of } \pm$$

$$= \exp[ik(\pm x \pm ct)]$$

Can show any function $U(x \pm ct)$ is a solution by change of variables

$$\xi = x - ct \quad \eta = x + ct \quad \Big\} \quad x = \tfrac{1}{2}(\eta + \xi) \quad t = \tfrac{1}{2c}(\eta - \xi)$$

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t}$$

and $\dfrac{\partial}{\partial \eta} = \dfrac{1}{2}\dfrac{\partial}{\partial x} + \dfrac{1}{2c}\dfrac{\partial}{\partial t}$

so $$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{1}{4c^2}\left(c\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

$$= \frac{1}{4c^2}\left(c^2\frac{\partial}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)$$

So in new variables, wave equation is

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = 0$$

so $U = A(\xi) + B(\eta)$ is a solution for any functions A, B

$x - ct$ right-propagating wave

$x + ct$ left-propagating wave

waves on finite string with fixed ends
$0 < x < L$; Boundary conditions $U(0,t) = U(L,t) = 0$

$u(x,t) = g(x) \cos(\omega t - \delta)$?

applying boundary conditions at $x=0$, $x=L$

$$g(x) = \sin(k_n x) \quad \text{with } k_n = \frac{n\pi}{L}$$

n is an integer

and $\omega_n = c k_n = \frac{c}{L}(n\pi)$

$n=1$ "fundamental"

$n=2$ "first harmonic"

General solution

$$U(x,t) = \sum_{n=1}^{\infty} D_n \sin(k_n x) \cos(\omega_n t - \delta_n)$$

$$= \sum_{n=1}^{\infty} \sin(k_n x) \left[B_n \cos \omega_n t + C_n \sin \omega_n t \right]$$

note, if we shift origin to $-\frac{L}{2} < x' < \frac{L}{2}$
we would have spatially symmetric and
anti-symmetric $g(x)$:

$\cos(k_n x')$ $n=1,3,5$ symmetric in space

$\sin(k_n x')$ $n=2,4,6$ antisymmetric in space

Orthogonality: $\int_0^L \sin(k_n x)\sin(k_m x)\,dx = \frac{L}{2}\delta_{nm}$

<u>initial conditions</u> determine B_N, C_N

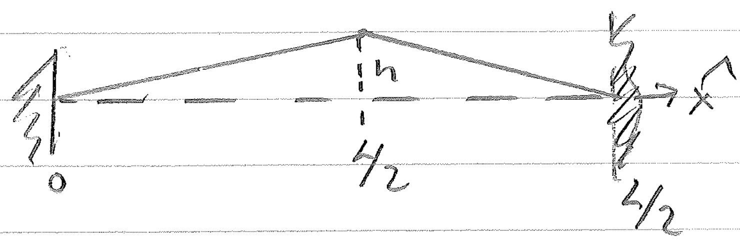
$$U(x,0) = \sum B_N \sin k_N x$$

$$\dot{U}(x,0) = \sum_N \omega_N C_N \sin k_N x$$

as $\frac{2}{L}\int_0^L dx \sin(k_N x)\, U(x,0) = B_N$

$$\frac{2}{L}\int_0^L dx \sin(k_N x)\, \dot{U}(x,0) = \omega_N C_N$$

<u>example</u>: plucked string



$$U(x,0) = \begin{cases} x\left(\frac{2h}{L}\right) & 0 < x < L/2 \\ \frac{2h}{L}(L-x) & \frac{L}{2} < x < L \end{cases}$$

$\dot{U}(x,0) = 0$ gives $C_N = 0$

$$B_n = \frac{2}{L}\int_0^{L/2} dx \left(\frac{2h}{L}\right) x \sin\left(\frac{n\pi x}{L}\right)$$

$$+ \frac{2}{L}\int_{L/2}^{L} dx \left(\frac{2h}{L}\right)(L - x)\sin\left(\frac{n\pi x}{L}\right)$$

$$= 2\left(\frac{2}{L}\right)\left(\frac{2h}{L}\right)\left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi/2} y \sin y \, dy$$

$$= \frac{8h}{n^2\pi^2}\left(\sin y - y\cos y\right)\Big|_0^{n\pi/2}$$

$$= \begin{cases} 0 & n \text{ even, spatially odd} \\ \dfrac{8h}{n^2\pi^2}(-1)^{\left(\frac{n-1}{2}\right)} & n \text{ odd, spatially even} \end{cases}$$

$$U(x,t) = \frac{8h}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos(3\omega_1 t) + \ldots\right]$$

$\omega_1 = \dfrac{\pi C}{L}$ fundamental