Physics 304 Spring 2019

Lec 16: Waves on a string

Energy of finite string: $L, \lambda, \overset{\text{tension}}{\tau}, c = \sqrt{\frac{\tau}{\lambda}}$ speed

λ mass/length

normal modes r: $k_r = \frac{r\pi}{L}$, $\omega_r = c k_r$

General solution, sum of normal modes (linearity of wave equation)

amplitude of mode

$$u(x,t) = \sum_{r=1}^{\infty} a_r \sin(k_r x) \cos(\omega_r t - \delta_r)$$

$$\dot{u}(x,t) = \sum_{r=1}^{\infty} -\omega_r a_r \sin(k_r x) \sin(\omega_r t - \delta_r)$$

Kinetic energy $T = \frac{\lambda}{2} \int_0^L (\dot{u})^2 dx$

$$\dot{u}^2 = \sum_{r,s} a_r a_s \omega_r \omega_s \sin k_r x \sin k_s x \sin(\omega_r t - \delta_r) \sin(\omega_s t - \delta_s)$$

orthogonality $\int_0^L dx \sin k_r x \sin k_s x = \frac{L}{2} \delta_{rs}$

giving

$$T = \frac{\lambda L}{4} \sum_{r=1}^{r_{max}} a_r^2 \omega_r^2 \sin^2(\omega_r t - \delta_r)$$

r_{max} is maximum $\omega_{r_{max}}$ that will not break the string.

Similarly for potential energy U *

$$U = \frac{\tau}{2}\int_0^L (u')^2 dx = \frac{L\tau}{4}\sum_{r=1}^{r_{max}} a_r^2 k_r^2 \cos^2(\omega_r t - \delta_r)$$

with $\tau = c^2\lambda$ and $\omega_r^2 = c^2 k_r^2$

$$E = T + U = \frac{1}{4}(L\lambda)\sum_{r=1}^{r_{max}} a_r^2 \omega_r^2 \quad \underline{\text{constant}}$$

$E_r \propto \omega_r^2$

Time average $\langle T \rangle = \frac{\omega_1}{2\pi}\int_0^{2\pi/\omega_1} T(t)\,dt$

where ω_1 is fundamental (lowest) frequency

with $\langle \sin^2 \rangle = \langle \cos^2 \rangle = \frac{1}{2}$

$$\langle T \rangle = \frac{L\lambda}{8}\sum_{r=1}^{\infty} a_r^2 \omega_r^2 = \langle U \rangle$$

* from loaded string $U = \frac{1}{2}\frac{T}{d}\sum_{j=1}^{N}(q_{j-1} - q_j)^2$

$$= \frac{T}{2}\sum_{j=1}^{N}\left(\frac{q_{j-1}-q_j}{d}\right)^2 d \xrightarrow[d\to 0]{} \frac{T}{2}\int (u')^2 dx$$

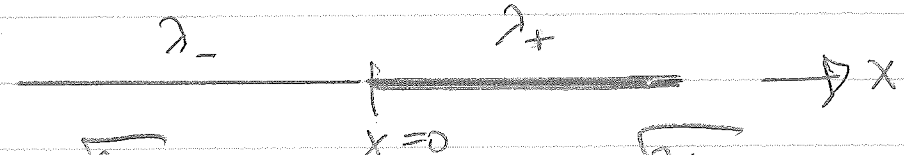
Transmission, Reflection at Boundary

"plane wave" $e^{i(kx-\omega t)}$ moves in +x direction

wave fronts (constant amplitude) are planes in 3 dimensions.

comment on wave packet

Plane wave incident from left $(-x)$ on very long string with two mass densities



$c_- = \sqrt{\tau/\lambda_-}$ $\qquad$ $c_+ = \sqrt{\tau/\lambda_+}$

for any frequency, we have $k_\pm = \omega/c_\pm = \omega\sqrt{\frac{\lambda_\pm}{\tau}}$

incident wave $u_{in} = e^{i\omega\left(\frac{x}{c_-} - t\right)}$

reflected and transmitted waves:

$u_r = B e^{i\omega\left(\frac{-x}{c_-} - t\right)}$ $\qquad$ $u_t = C e^{i\omega\left(\frac{x}{c_+} - t\right)}$

moves in −x direction $\qquad$ +x direction

total wave is

$$u(x,t) = \begin{cases} u_- = e^{i\omega\left(\frac{x}{c_-} - t\right)} + B e^{i\omega\left(-\frac{x}{c_-} - t\right)} \\ u_+ = C e^{i\left(\frac{x}{c_+} - t\right)} \end{cases}$$

Constants determined by boundary condition at $x=0$

① $u_-(0,t) = u_+(0,t)$ continuous

② $u_-'(0,t) = u_+'(0,t)$ continuous slope. no kink with $k_\pm = \omega / c_\pm$

① $1 + B = C$

② $ik_-(1-B) = ik_+ C$

$$B = C - 1 = \frac{k_-}{k_+}(1-B) - 1$$

$$B = \frac{k_- - k_+}{k_- + k_+} \qquad C = \frac{2k_-}{k_- + k_+}$$

Reflection coefficient $R = |B|^2$

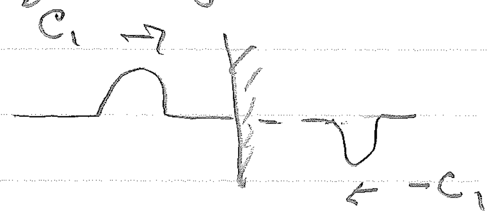
Transmission coefficient $T = 1 - R = \frac{k_+}{k_-}|C|^2$
in general, B, C complex (see thus #8.8)

consider limit $\frac{k_-}{k_+} = \sqrt{\frac{\lambda_-}{\lambda_+}} \ll 1$

$B \to -1 \qquad R = |B| \to 1 \quad , T \to 0$

<u>reflected wave is inverted.</u>

Very heavy string is like fixed boundary



"image wave"

This "method of images" follows from <u>uniqueness</u> of solution to wave equation

Phase & Group velocities

recall beats (see recitation)

superimpose 2 frequencies ω_1, ω_2

with $k_{1/2} = \omega_{1/2}/c$ wave number

$$u(x,t) = A\left[\cos(k_1 - \omega_1 t) + \cos(k_2 - \omega_2 t)\right]$$

suppose $\underbrace{\Delta k \equiv \frac{k_1 - k_2}{2}}_{\text{difference}} \ll \bar{k} \equiv \frac{k_1 + k_2}{2}$ average

with $\Delta\omega = c\,\Delta k$, $\bar{\omega} \equiv c\bar{k}$

then $u(x,t) = 2A \cos(\bar{k}x - \bar{\omega}t)\cos(\Delta k x - \Delta\omega t)$

$\rightarrow V_{group} = \frac{\Delta\omega}{\Delta k}$

$\rightarrow V_{phase} = \frac{\bar{\omega}}{\bar{k}}$

Phase velocity

Write $u(x,t) = a e^{i(kx - \omega t)}$ phase $\phi = kx - \omega t$

moving with plane wave, constant phase

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial t}dt = k\,dx - \omega\,dt = 0$$

phase velocity is

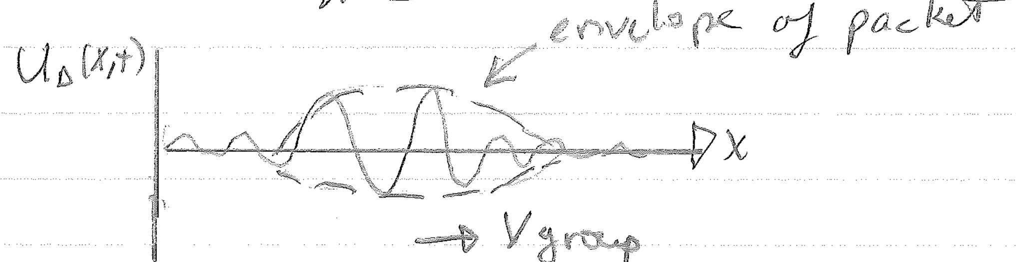
$$\frac{dx}{dt} = \frac{\omega}{k} = c = \sqrt{\frac{T}{\lambda}}$$

Wave Packet (term introduced by Schrodinger)

$$u(x,t) = \int dk\, a(k)\, e^{i(kx - \omega(k)t)}$$

for well defined wave packet $a(k) \cong 0$ outside of range $\bar{k} - \Delta < k < \bar{k} + \Delta$ and

$$U_\Delta(x,t) \cong \int_{\bar{k}-\Delta}^{\bar{k}+\Delta} a(k)\, e^{i(kx - \omega(k)t)}$$



$\omega(k)$ is called dispersion relation.

$$\text{expand} \quad \omega(k) \cong \underbrace{\omega(\bar{k})}_{\omega_0} + \underbrace{\frac{d\omega}{dk}\Big|_{\bar{k}}}_{\omega'} (k - \bar{k})$$

write phase as

$$kx - \omega t = (\bar{k} + k - \bar{k})x - (\omega_0 + \omega - \omega_0)t$$
$$= k_0 x - \omega_0 t + (k - \bar{k})x - (\omega - \omega_0)t$$
$$\cong k_0 x - \omega_0 t + (k - \bar{k})(x - \omega' t)$$

integral for wave packet

$$U_\Delta(x,t) = \mathrm{Re}\; e^{i(k_0 x - \omega t)} \int_{\bar{k}-\Delta}^{\bar{k}+\Delta} dk\, a(k)\, e^{i(k-k_0)(x-\omega' t)}$$

$$\underbrace{}_{f(x-\omega' t)}$$

Packet moves with group velocity given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt = dx - \omega' dt = 0$$

$$V_{group} = \frac{dx}{dt} = \omega' = \frac{d\omega}{\partial k}\bigg|_{\bar{k}}$$

for $V_g \neq V_{phase}$, wave packet will broaden (disperse) over time. Hence name for $\omega(k)$ as dispersion relation.

Wave equation in Three Dimensions

amplitude $U(\vec{r},t)$

equation $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \nabla^2 U$

where $\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial r_i^2}$ in Cartesian coordinates.

separation of variables:

$$U(\vec{r},t) = X(r_1,t)\,Y(r_2,t)\,Z(r_3,t)$$

with $X = e^{i(\pm k_x r_1 \pm \omega t)}$ etc. and so

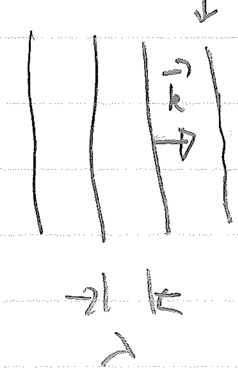
$$U_{\vec{k}}(\vec{r},t) = e^{i(\pm\vec{k}\cdot\vec{r} - \omega t)}$$

plane wave solutions

$\nabla^2 U_{\vec{k}}(\vec{r},t) = -k^2 U_{\vec{k}}$ and $\omega^2 = k^2c^2$

$$k^2 \equiv \vec{k}\cdot\vec{k}$$

$|\vec{k}| = \frac{2\pi}{\lambda}$ where λ is now <u>wavelength</u>



plane wave-fronts of constant U $\perp$ to direction of wave propagation $\vec{k}$.

General solution is superposition of plane waves.

spherical wave solution $U(r,\theta,\phi,t)$

look up ∇^2 in spherical coordinates

$$\nabla^2 = \nabla_r^2 + \frac{1}{r^2}\nabla^2_{\theta,\phi}$$

angular derivatives

can show $\nabla_r^2 = \left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r\right)$

Separating variables: $U = \frac{g(r)}{r} Y(\theta,\phi)\, f(t)$

spherical harmonics

$$f(t) = e^{\pm i\omega t} \quad \text{with } \omega = c|\vec{k}|$$

so $\nabla^2 U = -k^2 U$

$$Y \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r \cdot \frac{g}{r}\right) + \frac{1}{r^2}\frac{g}{r}\nabla^2_{\theta,\phi} Y = -k^2 \frac{g}{r} Y$$

$$\frac{1}{g}\frac{d^2}{dr^2}g + \underbrace{\frac{1}{r^2}\frac{1}{Y}\nabla^2_{\theta,\phi} Y}_{\text{another separation constant} \equiv -\Lambda} = -k^2$$

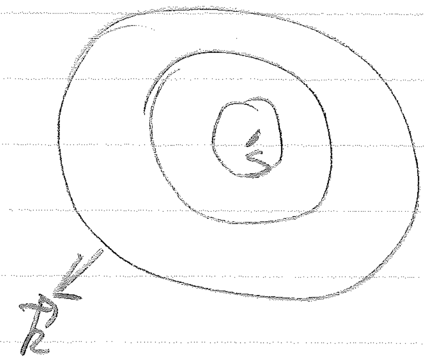
Spherically symmetric solutions have $\Lambda = 0$
for these, $\frac{d^2 g}{dr^2} = -k^2 g$

$$g(r) = e^{\mp ikr}$$

define $\vec{k} \parallel \vec{r}$ so $kr = \vec{k}\cdot\vec{r}$ then
spherical ($\Lambda = 0$) solutions are

$$U_s = \frac{1}{r} e^{i(\pm \vec{k}\cdot\vec{r} \pm \omega t)}$$

outgoing wave from point source



wave fronts are spheres separated by

$$\lambda = \frac{2\pi}{|\vec{k}|}$$

amplitude $\propto \frac{1}{r}$

flux $\propto \frac{1}{r^2}$ so Area $\times$ flux $=$ constant energy/time from source

$\frac{\text{energy}}{\text{area} \cdot \text{time}}$

spherical pulse from superposition are

$$U(\vec{r},t) = \frac{g(\vec{k}\cdot\vec{r} - \omega t)}{r}$$