Spring 2018

Physics 304

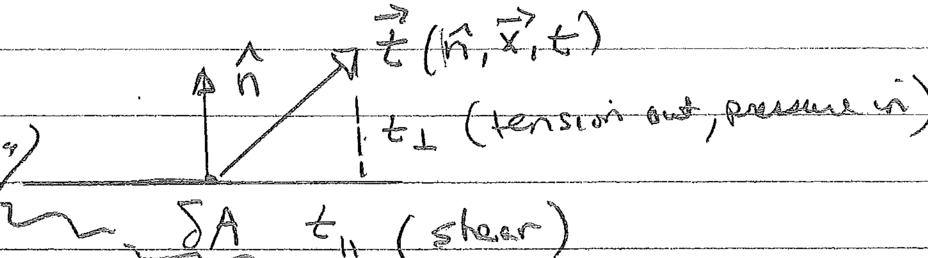
Lecture 18: Stress and Strain

Surface forces arise from intermolecular forces, so in continuum hypothesis are contact forces between neighboring points of material.
By Newton III, cancel except on surface.

Total stress on surface or traction (force/area)

$\vec{F}_s = \vec{t}\,\delta A$

("infinitesimal" area in continuum hypothesis sense)



how does $\vec{t}$ depend on $\hat{n}$?
traction $\vec{t}$ is defined as force on surface on material on $-\hat{n}$ side by material on $\hat{n}$ side.
So clearly,

$$\vec{t}(-\hat{n}, \vec{x}, t) = -\vec{t}(\hat{n}, \vec{x}, t)$$

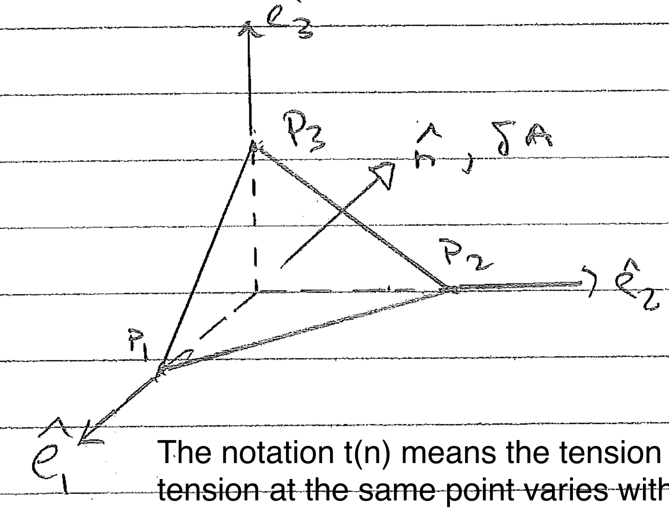
Newton's law for volume of material:

$$\int_V dV\, \rho\, \ddot{\vec{U}}(x,t) = \int_V \underbrace{\rho \vec{g}}_{\text{force/volume, body force}}\, dV + \int_{\partial V} (\text{surface forces})$$

Need vector from $\int_{\partial V}$ (surface forces)

Cauchy's stress theorem: For infinitesimal volume, ignore dependence of $\vec{t}$ on $\vec{x}$

Arbitrarily oriented triangular plane (P_1, P_2, P_3)
Orthogonal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$, area δA



The notation t(n) means the tension vector in the n-hat direction, tension at the same point varies with direction, so t(n)

$$\int_{\partial V} (\text{surface forces}) = \vec{t}(\hat{n})\,\delta A + \vec{t}(-\hat{e}_1)\,\delta A_1 + \vec{t}(-\hat{e}_2)\,\delta A_2 + \vec{t}(-\hat{e}_3)\,\delta A_3$$

minus signs for outward pointing normals.

$$\delta A_1 = \hat{e}_1 \cdot \hat{n}\,\delta A = n_1\,\delta A$$

$$\int_{\partial V} (\text{surface forces}) = \left\{ \vec{t}(\hat{n}) - \sum_{j=1}^{3} n_j \vec{t}(\hat{e}_j) \right\} \delta A$$

as $V \to 0$, $\delta A \to 0$ slower than V, so to satisfy Newton III, in limit in equilibrium

$$\vec{t}(\hat{n}) = \sum_{j=1}^{3} n_j \vec{t}(\hat{e}_j)$$

$V = \ell^3 \to \left(\frac{\ell}{2}\right)^3 = \frac{1}{8} V$

$A = \ell^2 \to \frac{1}{4} A$

In component form, $\vec{t}(\hat{n}) = \sum_{ij} \hat{e}_i \sigma_{ij} n_j$

$$t_i(\hat{n}) = \sum_{j=1}^{3} \underbrace{t_i(\hat{e}_j)}_{\equiv \sigma_{ij}} n_j$$

or $\vec{t} = \overline{\overline{\sigma}} \cdot \hat{n}$ since $\vec{t}$, $\hat{n}$ are vectors, $\overline{\sigma}$ is tensor under rotation group SO(3).

Now Cauchy didn't have vector calculus, but Landau and Lifshitz Theory of Elasticity do.

Recall Gauss's law in electrostatics –

$$\frac{1}{\varepsilon_0}\int_V \underbrace{\rho_q}_{\text{charge density}} dV = \int_V \underbrace{\vec{\nabla}\cdot\vec{E}}\, dV = \int_{\partial V} \underbrace{\vec{E}\cdot d\vec{a}}$$

$\vec{E} = -\vec{\nabla}\phi$

scalar $\longrightarrow$ vector

here we start with vector $\rightarrow$ tensor

Generalization of divergence theorem

$\vec{f} \equiv$ force/volume

$$\int_V \vec{f}\, dV = \oint_{\partial V} \overline{\sigma}\cdot d\vec{a} = \oint_{\partial V} \vec{t}\, da$$

vector $\longrightarrow$ tensor

or $f_i = \sum_j \partial_j \sigma_{ij}$ watch indicies!

surface

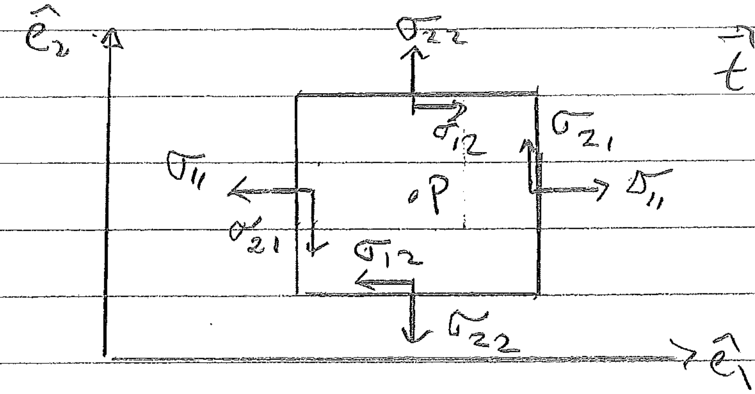
force

$$\boxed{t_i(\hat{n}) = \sum_j \sigma_{ij} n_j}$$

Stress tensor is symmetric.

As we defined it σ_{ij} ← surface direction
↳ force direction

Components on rectangular volume of square cross section (λ^2 by d) centered at point P



$\vec{t}(-\hat{n}) = -\vec{t}(\hat{n})$

$t_i(\hat{e}_j) = \sigma_{ij}$

arrows indicate direction of surface force for $\sigma_{ij} > 0$

torque about p in $\hat{e}_3$ –

$$\Gamma_3 = 2\left(\frac{\lambda}{2}\right)\lambda^2\left(\sigma_{21} - \sigma_{12}\right) \quad \propto \lambda^3$$

$$I\ddot{\phi} = \text{density} \times \underbrace{\text{volume}}_{\lambda^2 d} \times \left(\frac{\lambda}{2}\right)^2 \quad \propto \lambda^4$$

So by Newton III in limit $\lambda \to 0$,

$$\boxed{\sigma_{21} = \sigma_{12}} \quad \text{symmetric}$$

So we have $\vec{f} = \vec{\nabla} \cdot \bar{\boldsymbol{\sigma}}$ in vector notation

Stress tensor for fluid

In static fluid, shear is zero. Also true for zero viscosity (" inviscid ") fluid because shear results from viscosity.

Pressure in static fluid is isotropic (similar to previous argument for symmetry, see text)

$$\sigma_{ij} = -P\delta_{ij} \qquad \text{(static fluid)}$$

P = pressure

In general, non-static

$$\sigma_{ij} = \underbrace{-P\delta_{ij}}_{\text{static}} + \overbrace{d_{ij}}^{}\ \ \text{motion, symmetric}$$

d_{ij} depends on viscosity and can be expressed as derivatives of fluid velocity field:

$$\vec{V}(\vec{x},t)$$

$$d_{ij} = \eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) + \left(\eta' - \tfrac{2}{3}\eta\right)\left(\sum_{k=1}^{3}\frac{\partial V_k}{\partial x_k}\right)\delta_{ij}$$

η = viscosity, η' = second viscosity

Incompressible fluid has $\eta' - \frac{2}{3}\eta = 0$

Water very nearly incompressible

Principle Stress Axes

$\bar{\sigma}$ can always be diagonalized at a point.

Useful in case of plane boundary.

Strain deformation under stress

Displacement vector field $\vec{U}(\vec{r})$

$$\vec{r}(t) \rightarrow \vec{r}(t) + \vec{U}(\vec{r}, t)$$

Consider only small deformations of solids.

Taylor "removes" irrelevant translations and rotations. I follow more elegant & physical Landau & Lifshitz approach.

Consider distance between two nearby points:

$$(d\vec{l})^2 = \sum_{i=1}^{3} (dx_i)^2 \quad \text{by Pythagoras.}$$

$$\vec{r} \rightarrow \vec{r}' = \vec{r} + \vec{U} \qquad dx_i \rightarrow dx_i' = dx_i + du_i$$

$(d\vec{l}')^2 = \sum_i (dx_i + du_i)^2$ for small displacements, ignore $(du_i)^2$ term to get

$(d\vec{l}')^2 = (d\vec{l})^2 + 2\sum_i dx_i du_i$

by chain rule, $du_i = \sum_j \frac{\partial u_i}{\partial x_j} dx_j$

2nd term is

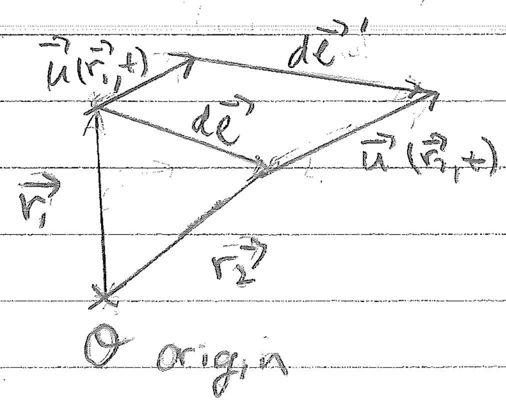
$$2\sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right) dx_i dx_j$$

$$= 2\sum_{i,j} \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{Symmetric } E_{ij} \text{ strain tensor.}} dx_i dx_j$$

here I made use of I,j dummy indices to write as tensor

$(d\vec{l}')^2 = (d\vec{l})^2 + 2\, d\vec{l} \cdot \bar{E} \cdot d\vec{l}$

$d\vec{l} = \sum_{i=1}^{3} dx_i \hat{e}_i$



Examples of strain

① dilation (pure expansion)

every point $\vec{r} \to \vec{r}' = (1+e)\vec{r}$

volume $V = \vec{a} \cdot (\vec{b} \times \vec{c}) \to V' = \vec{a}' \cdot (\vec{b}' \times \vec{c}')$

$$= (1+e)\vec{a} \cdot \left((1+e)\vec{b} \times (1+e)\vec{c} \right)$$
$$\cong \vec{a} \cdot (\vec{b} \times \vec{c}) \; (1 + 3e)$$

$$\Delta V = V' - V = 3eV \qquad \frac{\Delta V}{V} = 3e$$

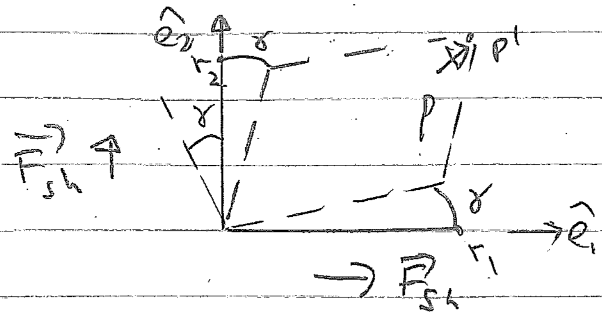
$$\vec{r}' \cdot \vec{r}' = \left((1+e)\vec{r} \right) \cdot \left((1+e)\vec{r} \right) \cong r^2 + 2e\,\vec{r} \cdot \vec{r}$$

$$= |\vec{r}|^2 + 2\,\vec{r} \cdot \bar{\bar{\varepsilon}} \cdot \vec{r}$$

$$\bar{\bar{\varepsilon}} = e\,\bar{\bar{1}} \qquad (\bar{\bar{1}})_{ij} = \delta_{ij}$$

unit matrix

② <u>Shearing (pure shear)</u>



ignore $\hat{e}_3$ direction for which there is no deformation

apply symmetric shear $\quad \vec{r}_p \to \vec{r}_p\,'$

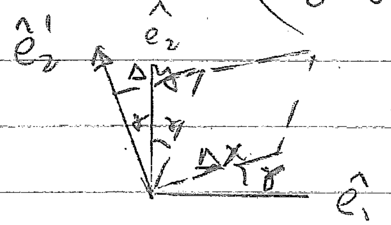
$$\vec{r}_p\,' = (r_1\cos\gamma + r_2\sin\gamma)\,\hat{e}_1 + (r_1\sin\gamma + r_2\cos\gamma)\,\hat{e}_2$$

$$\vec{r}_p\,'\cdot\vec{r}_p\,' = r_1^2 + r_2^2 + 4 r_1 r_2 \cos\gamma\sin\gamma \underset{\text{small } \gamma}{\approx} r_1^2 + r_2^2 + 4\gamma r_1 r_2$$

$$= \vec{r}\cdot\vec{r} + 2\,(r_1, r_2)\begin{pmatrix}0 & \gamma\\ \gamma & 0\end{pmatrix}\begin{pmatrix}r_1\\ r_2\end{pmatrix}$$

$$= \vec{r}\cdot\vec{r} + 2\,\vec{r}\cdot\bar{\bar{E}}\cdot\vec{r}$$

$$\bar{\bar{E}} = \begin{pmatrix}0 & \gamma & 0\\ \gamma & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad \text{pure shear}$$



box of side Δx

$$\frac{\Delta y}{\Delta x} = \tan(2\gamma) \cong 2\gamma$$

<u>engineering strain</u> is 2γ

<u>Note</u>: volume unchanged – pure shear is traceless