Physics 304 Spring 2015

Lecture 19: Relation of Stress & Strain

Can show that M_Y, M_B, M_S are determined by only 2 properties of material (α, β).

$$\bar{\sigma} = f(\bar{E})$$ where f is some rotationally invariant function.

formally, $f(\bar{R}\bar{E}\bar{R}^T) = \bar{R}\bar{\sigma}\bar{R}^T$

$\bar{E}$ can be decomposed into two pieces that do not mix under rotations (called irreducible in group theory)

E is reducible into unit matrix plus E'

E' is irreducible

$$\bar{E} = e\bar{1} + \bar{E}' \quad \text{where } tr(\bar{E}') = 0$$
$$\text{so } e = \tfrac{1}{3} tr(\bar{E})$$

Thus, most general rotationally invariant relation is:

$$\boxed{\bar{\sigma} = \alpha(e\bar{1}) + \beta\bar{E}' = (\alpha-\beta)e\bar{1} + \beta\bar{E}}$$

invert to get $\bar{E} = f^{-1}(\bar{\sigma})$

$$tr(\bar{\sigma}) = 3\alpha e$$

$$\bar{\sigma} = \frac{(\alpha-\beta)}{3\alpha}(tr\,\bar{\sigma})\,\bar{1} + \beta\bar{E}$$

gives:

$$\boxed{\begin{aligned}\bar{E} &= -\left(\frac{\alpha-\beta}{3\alpha\beta}\right)(tr\,\bar{\sigma})\,\bar{1} + \frac{1}{\beta}\bar{\sigma} \\ &= \left(\frac{1}{3\alpha\beta}\right)\left[(\beta-\alpha)(tr\,\bar{\sigma})\,\bar{1} + 3\alpha\bar{\sigma}\right]\end{aligned}}$$

Bulk Modulus: pure dilation, no shear

$$\bar{\sigma} = -p\bar{1}, \quad tr(\bar{\sigma}) = -3p$$

$$\bar{E} = \frac{1}{3\alpha\beta}\left[-3\alpha p\,\bar{1} - (\alpha-\beta)(-3p)\,\bar{1}\right] = \frac{-p}{\alpha}\bar{1}$$

or $e = -p/\alpha$ with $\bar{E}' = 0$

$$V \to (1+e)^3 V \cong (1+3e)V = V + \Delta V$$

$$\frac{\Delta V}{V} = 3e = -\frac{3}{\alpha}p$$

$$\boxed{\frac{-p}{\Delta V/V} = M_B = \frac{\alpha}{3}}$$

"constant ratio"

<u>Shear modulus</u> pure shear $e = 0$ or $\mathrm{tr}\,\bar{E} = 0$

from example $\bar{E} = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $2\gamma = \frac{dy}{dx}$

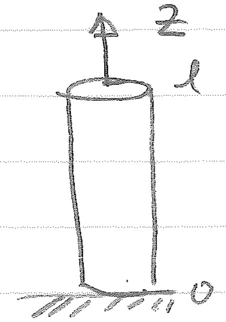
see remark p. 7.17

$\bar{\sigma} = \beta \bar{E}$ $\sigma_{12} = \text{stress} = \frac{\text{force}}{\text{area}} = \beta\gamma$

$$\boxed{\frac{\text{stress}}{\text{deformation}} = m_s = \frac{\beta\gamma}{2\gamma} = \frac{\beta}{2}}$$

REDACTED

Example: compression of standing rod
Here we consider only stress along gravitational direction (ignore transverse deformation).



$$t_z = -\rho g\,(l - z) = \sigma_{zz}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{1}{M_y}\sigma_{zz} = -\frac{\rho g}{M_y}(l - z)$$

$$u(z) = \frac{-\rho g}{M_y}\left(lz - \frac{1}{2}z^2 + \underset{0}{\text{const}}\right)$$ define displacement to be zero at z=0

$$\Delta l = |u(l)| = \frac{\rho g}{2M_y}\,l^2$$

$$\frac{\Delta l}{l} = \frac{l}{[2M_y/\rho g] \equiv D}$$

$$D|_{steel} = \frac{2(200\times10^9\ N/m^2)}{8\times10^3\,kg/m^3 \cdot 10\ m/s^2} = \frac{1}{2}\times10^7\ m$$ (Pa)

For building $\rho = \frac{1}{10}\rho_w = 100\ kg/m^3$ = density of steel/80
and steel supports with $f = 1\%$ of area
fill factor

$$D|_{building} = \left(80\cdot\frac{1}{100}\right)D_{steel} = 4000\ km$$

tallest building $l \approx .8\ km$, $\Delta l = \frac{(.8\,km)^2}{4000\,km} = \underline{16\ cm}$

Equation of motion for elastic solid with gravity (writing $\ddot{\vec{U}} = \frac{\partial^2 \vec{U}}{\partial t^2} = \vec{U}_{tt}$

$$\int_V \rho\, dV\, \vec{U}_{tt} = \int_V \rho \vec{g}\, dV + \int_{\partial V} \bar{\sigma} \cdot d\vec{a}$$

using divergence theorem $\left(\int_{\partial V} \bar{\sigma} \cdot d\vec{a}\right)_i = \sum_j \int_V \frac{\partial \sigma_{ij}}{\partial x_j}\, dV$

$$\rho \ddot{U}_i = \rho g_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} = \rho g_i + \left(\vec{\nabla} \cdot \bar{\sigma}\right)_i$$

using symmetry of $\bar{\sigma}$ *

$$\sigma_{ij} = (\alpha - \beta)\, e\, \delta_{ij} + \beta E_{ij}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = \frac{1}{2}\left(\partial_j U_i + \partial_i U_j\right)$$

where $\partial_j \equiv \frac{\partial}{\partial x_j}$

dilation part come back to this

$$e = \frac{1}{3} \operatorname{tr}(\bar{E}) = \frac{1}{3} \sum_i \partial_i U_i = \frac{1}{3} \vec{\nabla} \cdot \vec{u}$$

* using symmetry

$$\sum_j \partial_j \sigma_{ij} = \left(\vec{\nabla} \cdot \bar{\sigma}^T\right)_i = \left(\vec{\nabla} \cdot \bar{\sigma}\right)_i$$

use $\bar{\sigma} = f(\bar{E})$ to write in terms of derivatives of $\vec{u}$

$$\sum_{j=1}^{3} \partial_j \sigma_{ij} = \sum_{j=1}^{3} (\alpha-\beta)\frac{\partial e}{\partial x_j}\delta_{ij}$$

$$+ \frac{\beta}{2}\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j\right)$$

$$= (\alpha-\beta)\partial_i e + \frac{\beta}{2}\left[\nabla^2 u_i + \partial_i(\vec{\nabla}\cdot\vec{u})\right]$$

using $\partial_i e = \frac{1}{3}\partial_i(\vec{\nabla}\cdot\vec{u})$ from previous page

and $\frac{\alpha-\beta}{3} + \frac{\beta}{2} = \frac{1}{3}\left(\alpha + \frac{\beta}{2}\right)$

$$(\vec{\nabla}\cdot\bar{\sigma})_i = \sum_{j=1}^{3}\partial_j \sigma_{ij} =$$ div u parts combine

$$\frac{1}{3}\left(\alpha+\frac{\beta}{2}\right)\partial_i(\vec{\nabla}\cdot\vec{u}) + \frac{\beta}{2}\nabla^2 u_i$$

or $\vec{\nabla}\cdot\bar{\sigma} = \frac{1}{3}\left(\alpha+\frac{\beta}{2}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \frac{\beta}{2}\nabla^2\vec{u}$

Gives Navier Equation

$$\rho\,\vec{u}_{tt} = \rho\,\vec{g} + \frac{1}{3}\left(\alpha+\frac{\beta}{2}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \frac{\beta}{2}\nabla^2\vec{u}$$

2nd order, linear P.D.E. for $\vec{u}(\vec{r},t)$

note that $\frac{1}{3}\left(\alpha+\frac{\beta}{2}\right) = M_B + \frac{1}{3}M_S$

and $\frac{\beta}{2} = M_S$

Longitudinal Plane Waves

plane wave propagating in x-direction with $g=0$

$$\vec{U} = \left(U_x(x-ct),\ 0,\ 0\right)$$

$$\vec{\nabla}\cdot\vec{U} = \partial_x U_x \qquad \nabla^2 \vec{U} = \partial_x^{\,2} U_x \,\hat{x}$$

same

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{U}) = \partial_x^{\,2} U_x \,\hat{x}$$

$$\vec{\nabla}\cdot\vec{\sigma} = \left(\frac{\alpha}{3} + \frac{\beta}{6} + \frac{\beta}{2}\right)\partial_x^{\,2} U_x \,\hat{x} = \rho\, \partial_t^{\,2} U_x \,\hat{x}$$

$\left(\frac{\partial^2}{\partial t^2}\right.$

$$\left(\frac{\alpha + 2\beta}{3}\right)\frac{\partial^2 U_x}{\partial x^2} = \rho \frac{\partial^2 U}{\partial t^2}$$

speed $V_\ell = \sqrt{\dfrac{\alpha + 2\beta}{3\rho}} = \sqrt{\dfrac{3M_B + 4M_S}{3\rho}}$

transverse plane waves

plane wave propagating in x-direction with amplitude (polarization) in y-direction

$$\vec{U} = (0, U_y(x-ct), 0)$$

$$\vec{\nabla}\cdot\vec{U} = \frac{\partial U_y}{\partial y} = 0$$

$$\nabla^2 \vec{U} = \frac{\partial^2 U_y}{\partial x^2}\,\hat{x}$$

$$\rho \frac{\partial^2 U_y}{\partial t^2} = \frac{\beta}{2}\frac{\partial^2 U_y}{\partial x^2}$$

speed $V_t = \sqrt{\dfrac{M_s}{\rho}}$

example 16.8 for granite

$M_B = 40\,\text{GPa}$ $\quad M_s = 25\,\text{GPa}$ $\quad \rho = 2.7\times10^3\,\text{kg/m}^3$

$V_\ell = 5.25\ \text{km/s}$

$V_t = 3.00\ \text{km/s}$