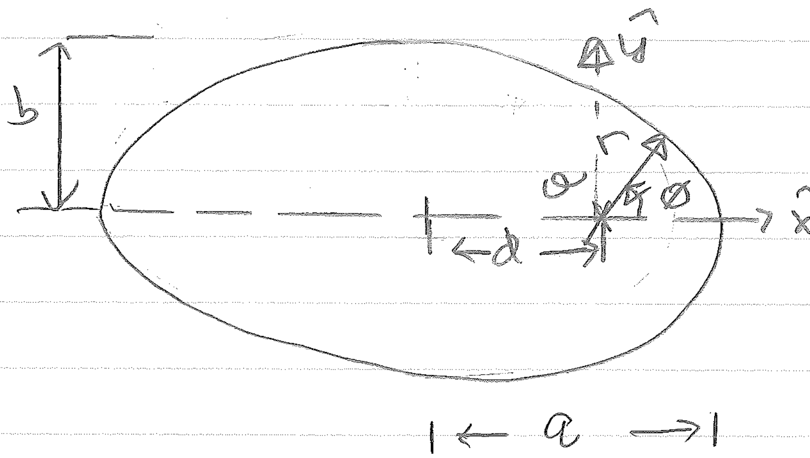
Lecture # 2: Kepler

Kepler I: $m_E \ll m_s$ $\quad \frac{m_E}{m_S} \approx 3\times10^{-6}$

$\mu \cong m_E$ and center of sun is $\approx$ C.M.
Bound orbits are ellipses with sun at one focus.



define $c \equiv \frac{\ell^2}{\gamma\mu}$ $\quad \gamma \equiv G m_s m$

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$ $\varepsilon \equiv$ eccentricity

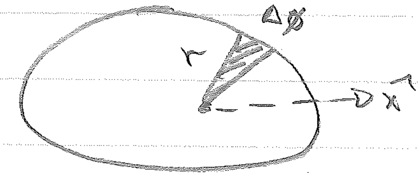
$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad d = a\varepsilon$$

and $a = \frac{c}{1-\varepsilon^2}$; $b = \frac{c}{\sqrt{1-\varepsilon^2}}$

$$b = \sqrt{ca}$$

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$

Kepler II: Conservation of $\ell = \mu r^2 \dot{\phi}$
orbital angular momentum.



area enclosed $\quad \Delta A = \frac{1}{2} r (r \Delta\phi)$

$$\frac{d}{dt} A = \frac{1}{2} r^2 \dot{\phi} = \frac{\ell}{2\mu} = \text{constant}$$

"equal areas in equal times"

Kepler III: $(\text{orbital period } \tau)^2 \propto a^2$

Total area of ellipse $\quad A = \pi a b = \frac{\ell}{2\mu} \tau$

$$\left(\frac{\ell}{2\mu} \tau\right)^2 = \pi^2 a^2 (c a)$$

$$\tau^2 = \left(\frac{2\mu}{\ell}\right)^2 \left(\frac{\ell^2}{\gamma\mu}\right) \pi^2 a^3 = \frac{4\pi^2 \mu}{\gamma} a^3$$

for $m_s >> m \quad \frac{\gamma}{\mu} \simeq G M_s$ and $\boxed{\tau^2 \propto a^3}$

Orbits: Most simply from energy conservation.

$$E = U_{eff}(r_{min}) = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}$$

change variable from t to ϕ -

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}$$

$$\frac{dr}{d\phi} = \frac{\mu r^2}{\ell}\sqrt{\frac{2}{\mu}(E-U(r)) - \frac{\ell^2}{\mu^2 r^2}}$$

Integrate

$$\phi(r) = \int \frac{\frac{\ell}{r^2}\,dr}{\sqrt{2\mu(E-U) - \frac{\ell^2}{r^2}}}$$

with $U = -\frac{\gamma}{r}$ and let $x = 1/r$

$$\phi(x) = -\int \frac{dx}{\left[\frac{2\mu E}{\ell^2} + 2\left(\frac{\mu\gamma}{\ell^2}\right)x - x^2\right]^{1/2}}$$

define $\boxed{C = \frac{\ell^2}{\mu\gamma}}$ where $[C]$ = length

Complete the square:

$$[\,] = -\left(x - \frac{1}{c}\right)^2 + \frac{1}{c^2}\Big(1 + \underbrace{\frac{2\mu c^2 E}{\ell^2}}_{=\frac{2E\ell^2}{\mu\gamma^2}}\Big)$$

define $\boxed{\varepsilon \equiv \left(1 + \frac{2E\ell^2}{\mu\gamma^2}\right)^{1/2}}$ then

$$\phi(x) = -\int \frac{dx}{\left[\frac{\varepsilon}{c^2} - \left(x - \frac{1}{c}\right)^2\right]^{1/2}}$$

$$= -\frac{c}{\varepsilon}\int \frac{dx}{\left[1 - \left(\frac{cx-1}{\varepsilon}\right)^2\right]^{1/2}}$$

let $y = \dfrac{cx-1}{\varepsilon}$

$$\phi(y) = -\int \frac{dy}{\sqrt{1-y^2}} \quad \left\}\begin{array}{l} y = \cos z \\ dy = -\sin z\, dz \\ \sqrt{1-y^2} = \sin z \end{array}\right.$$

$$\phi(x) = \cos^{-1}\left(\frac{cx-1}{\varepsilon}\right) + \underset{\equiv 0}{\text{const}}$$

$$\varepsilon \cos\phi = \frac{c}{r} - 1$$

$$\boxed{r(\phi) = \frac{c}{1 + \varepsilon\cos\phi}}$$ ellipse in polar coordinates

Alternatively, Lagrange equation

$$\mu \ddot{r} = \frac{\ell^2}{\mu r^3} - U' \quad , U' = F = \frac{-\gamma}{r^2}$$

let $x = 1/r$

$$\frac{dx}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\phi}} = \frac{-\mu}{\ell}\dot{r}$$

$\uparrow \dot{\phi} = \ell/\mu r^2$

$$\frac{d^2x}{d\phi^2} = \frac{d}{d\phi}\left(\frac{-\mu}{\ell}\dot{r}\right) = \frac{1}{\dot{\phi}}\frac{d}{dt}\left(\frac{-\mu}{\ell}\dot{r}\right)$$

$$= \frac{-\mu}{\ell\dot{\phi}}\ddot{r} = \frac{-\mu^2}{\ell^2}r^2\ddot{r}$$

so $$\mu r^2 \ddot{r} = \frac{-\ell^2}{\mu}\left(\frac{d^2x}{d\phi^2}\right)$$

eq. of motion is in variable X,

$$\frac{-\ell^2}{\mu}\frac{d^2X}{d\phi^2} = \frac{\ell^2}{\mu}X + \frac{F(1/X)}{X^2}$$

$$F(1/X) = -\gamma X^2 \quad \text{so}$$

$$\frac{d^2X}{d\phi^2} + X = \frac{\gamma\mu}{\ell^2} \equiv \frac{1}{c}$$

let $y = X - 1/c$

$$\frac{d^2y}{d\phi^2} = -y \quad \Rightarrow y(\phi) = A\cos(\phi - \delta)$$

choose $\delta = 0$

$$X(\phi) = A\cos(\phi) + \frac{1}{C}$$

Let $A = \frac{\varepsilon}{C}$

$$\frac{1}{r} = \frac{1}{C}\left(1 + \varepsilon\cos\phi\right)$$

$$r = \left(\frac{C}{1+\varepsilon\cos\phi}\right)$$

<u>note</u> - we don't have relation for eccentricity ε.