Physics 304 Spring 2018

Lecture 20: Fluids

Viscosity for common fluids (e.g. air, water) is "small" and can be neglected, particularly away from a boundary. Idealized, zero viscosity fluid called "inviscid".

Viscosity $[\eta]$ = pressure × time

unit $\frac{1}{10}$ Pa·s $\equiv$ 1 Poise ; m Pa·s = centipoise (cPs)

η(water) = 1 cPs

η(SAE 10 motor oil) = 85–140 cPs

Dynamical variable is fluid velocity field

$$V(\vec{r}, t)$$

Newton's second law concerns motion of "material point" or "parcel" of fluid. We must consider change with respect to parcel known as material derivative.

Two ways of thinking about fluid flow.

<u>Eulerian</u> $V(\vec{r},t)$ at points fixed in space
"watch the flow go by"

<u>Lagrangian</u> "go with the flow"
material derivative

Consider density (scalar field) $\rho(\vec{r},t)$

$$\Delta\rho \equiv \rho(\vec{r}+\Delta\vec{r},\ t+\Delta t) - \rho(\vec{r},t)$$

$$= \frac{\partial\rho}{\partial t}\Delta t + (\vec{\nabla}\rho)\cdot\Delta\vec{r}$$

limit $\Delta\vec{r}\to 0$, $\Delta t\to 0$ but $\frac{\Delta\vec{r}}{\Delta t}\to\vec{v}$

material derivative

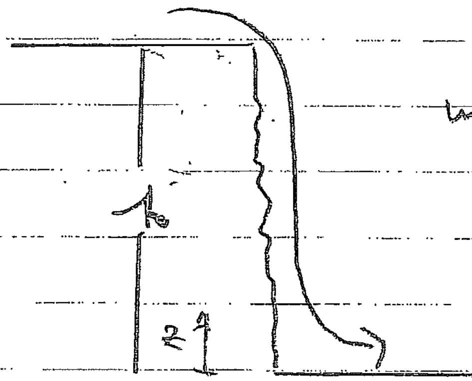
$$\frac{D\rho}{Dt} = \lim_{\substack{\Delta t\to 0\\ \frac{\Delta\vec{r}}{\Delta t}\to\vec{v}}} \frac{\Delta\rho}{\Delta t} = \frac{\partial\rho}{\partial t} + \vec{V}\cdot(\vec{\nabla}\rho)$$

Generally,

$$\frac{D}{Dt} = \vec{V}\cdot\vec{\nabla} + \frac{\partial}{\partial t}$$

Cartesian $\sum_{i=1}^{3} V_i \frac{\partial}{\partial x_i}$

Example water over the dam



Steady flow $V_z(z,t)$

with $\frac{\partial V_z}{\partial t} = 0$

ρ = constant

$$-g = \frac{DV_z}{Dt} = \underset{0}{\underset{\shortparallel}{\frac{\partial V_z}{\partial t}}} + \left(V_z \frac{\partial}{\partial z}\right) V_z$$

$$-g = \frac{1}{2} \frac{d}{dz} V_z^2$$

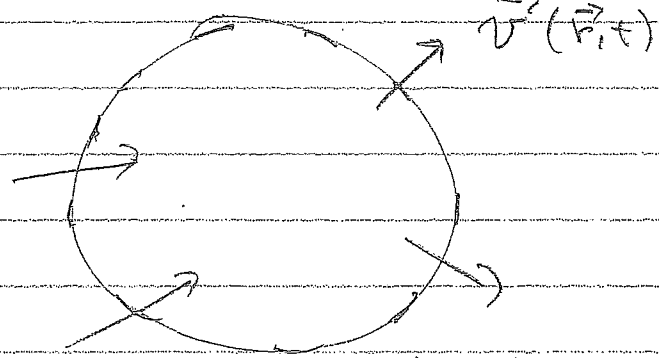
$$V_z^2 = 2g(h-z) \qquad [V_z(h) = 0]$$

$$V_z(z,t) = \sqrt{2g(h-z)}$$

corresponds to energy conservation $\frac{1}{2} m V(z)^2 = mg(h-z)$ for mass m

Equation of Continuity = conservation of mass

{Eulerian View}



V fixed in space

$$\frac{d}{dt}\int_V \rho\, d^3\vec{r} = -\int_{\partial V} \rho\,\vec{v}\cdot d\vec{a} = -\int_V \vec{\nabla}\cdot(\rho\vec{v})\,d^3\vec{r} = 0$$

$\frac{\partial}{\partial t}$ since V is constant

divergence theorem

thus $\quad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$

$$\vec{\nabla}\cdot(\rho\vec{v}) = (\vec{\nabla}\rho)\cdot\vec{v} + \rho(\vec{\nabla}\cdot\vec{v})$$

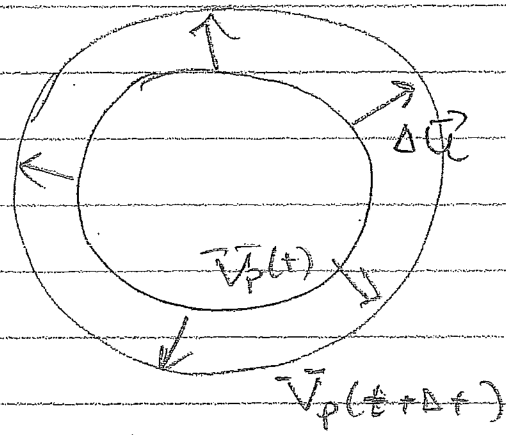
$$\frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho + \rho(\vec{\nabla}\cdot\vec{v}) = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho\,\vec{\nabla}\vec{v} = 0}$$

Note: incompressible fluid $\quad \frac{D\rho}{Dt} = 0 = \vec{\nabla}\cdot\vec{v}$

Lagrangian view

Change of volume of parcel $\overline{V}_p(t)$



volume expands due to displacement of material $\Delta\vec{u}$ in time Δt

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{u}}{\Delta t} \quad \text{then}$$

$$\frac{d\overline{V}_p}{dt} = \frac{D\overline{V}_p}{Dt} = \int_{\partial V_p} \vec{v}\cdot d\vec{a} = \int_{V_p} (\vec{\nabla}\cdot\vec{v})\, d^3\vec{r}$$

$\uparrow$ V_p depends only on time

in limit as $\overline{V}_p \to 0$ $\quad \int_{V_p} (\vec{\nabla}\cdot\vec{v})\, d^3\vec{r} = \vec{\nabla}\cdot\vec{v}\; \overline{V}_p$

giving

$$\frac{1}{\overline{V}_p}\frac{D\overline{V}_p}{Dt} = \vec{\nabla}\cdot\vec{v}$$

then $\boxed{\frac{D}{Dt}\left(\rho\overline{V}_p\right) = 0}$ continuity equation conservation of mass

$\boxed{\text{or } \rho\overline{V}_p = \rho_0 V_p(0)}$

Note $\vec{\nabla}\cdot\vec{v} = 0 = \frac{D\overline{V}_p}{Dt}$ Volume of incompressible fluid parcel does not change

How compressible is water?

$M_B = 2.2$ GPa for water

At bottom of Pacific (4000 m)

$P = 4 \times 10^7$ Pa $= 0.04$ GPa so

$$\frac{\Delta V}{V} \approx 2\%$$

<u>Streamlines:</u> Two dimensional, incompressible flow

$$\vec{v}(x,y) = (V_x, V_y, 0)$$

incompressible condition $\nabla \cdot \vec{v} = 0 = \partial_x V_x + \partial_y V_y = 0$

let ψ be scalar function with

$$V_x \equiv \frac{\partial \psi}{\partial y} \qquad V_y \equiv -\frac{\partial \psi}{\partial x} \qquad \text{then}$$

$$\vec{\nabla} \cdot \vec{v} = 0 = \partial_x \partial_y \psi - \partial_x \partial_y \psi = 0$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -V_y\, dx + V_x\, dy$$

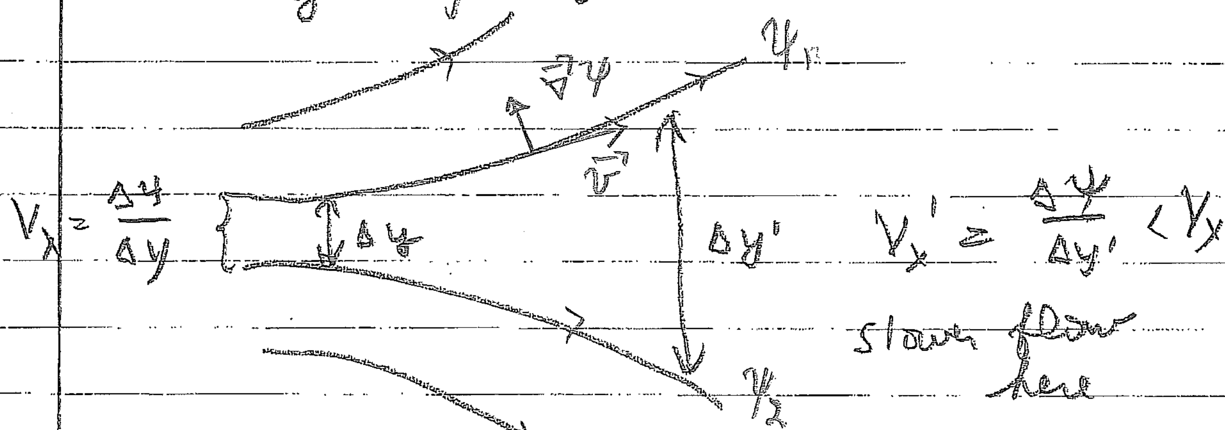
$$\Delta \psi_{AB} = \int_A^B (V_x\, dy - V_y\, dx)$$

Streamlines are lines of <u>constant</u> ψ $(\Delta\psi = 0)$

$\Delta\psi = 0 \Rightarrow V_x\,dy - V_y\,dx$

tangent to streamline $\dfrac{dy}{dx} = \dfrac{-\partial\psi/\partial x}{\partial\psi/\partial y} = \dfrac{V_y}{V_x}$

Example flow – draw streamlines at constant interval $\psi_1 - \psi_2 \equiv \Delta\psi$ for all neighboring lines



$\vec{\nabla}\psi \cdot \vec{v} = \underbrace{\dfrac{\partial\psi}{\partial x}}_{=-V_y} V_x + \underbrace{\dfrac{\partial\psi}{\partial y}}_{=+V_x} V_y = 0$

Irrotational flow — no closed streamlines

$$\underbrace{\oint_{\partial A}}_{\text{loop}} \vec{v}\cdot d\vec{s} = \underbrace{\int_A}_{\text{area}} (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$$

So there exists velocity potential function

$$\vec{v} = \vec{\nabla}\phi$$

$\partial_x \phi = V_x \quad \partial_y \phi = V_y$

<u>Incompressible & irrotational</u> 2D flow

$\vec{v} = \vec{\nabla}\phi$ potential

$$\int_{\text{streamline}} \vec{\nabla}\psi \cdot d\vec{s} = 0 \qquad \vec{\nabla}\psi = (-V_y, V_x)$$

$$(\vec{\nabla}\psi) \cdot \vec{\nabla}\phi = (-V_y, V_x)\begin{pmatrix} V_x \\ V_y \end{pmatrix} = 0$$

lines of constant potential orthogonal to streamlines.

We have: $V_x = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$

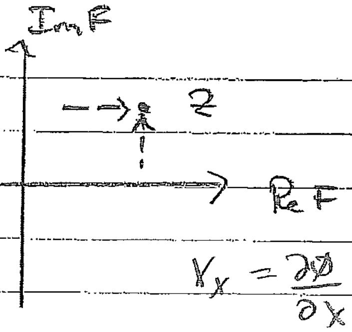
$$V_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

<u>Cauchy-Riemann equations</u> $z = x + iy$

$F(z) = \phi + i\psi$ complex, analytic potential function.

Analytic means $\frac{dF}{dz}$ independent of path to limit point.

Analytic function $F(z) = \phi + i\psi$



$$\left.\frac{dF}{dz}\right|_{\text{real},\ z=x} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

$$V_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$

$$V_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$$\left.\frac{dF}{dz}\right|_{\text{imaginary}\ z=iy} = \frac{1}{i}\left(\frac{\partial\phi}{\partial y} + i\frac{\partial\psi}{\partial y}\right) = \frac{\partial\psi}{\partial y} - i\frac{\partial\phi}{\partial y}$$

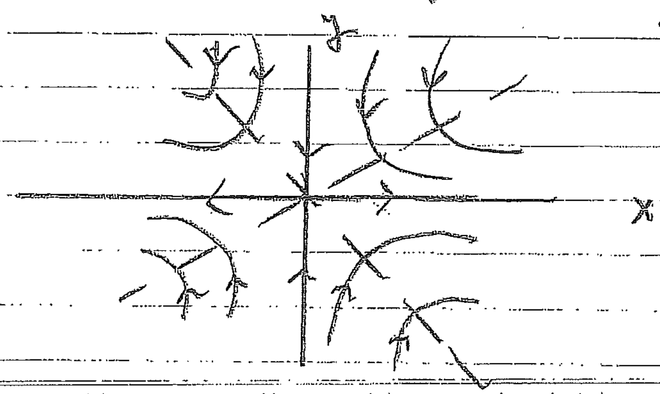
gives Cauchy-Rieman: $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$; $\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}$

$$f(z) = \frac{dF}{dz} = V_x - iV_y$$ also analytic

However, $f^*(z) = V_x + iV_y$ (complex velocity) is not analytic.

Example: Flow near stagnation point

$\vec{v}(0,0) = (0,0)$



$$F = \frac{1}{2}kz^2 = \frac{1}{2}k(x+iy)^2 = \frac{1}{2}k(x^2-y^2+2ixy)$$

$\phi = \frac{1}{2}k(x^2-y^2)$ Velocity potential

$\psi = kxy$ hyperbolae streamline function

See Schwartz-Christoffel transformation for applications to boundary value problems

Bernoulli's Theorem Steady flow of incompressible, inviscid ($\eta=0$) fluid.

External volume force is gravity $\vec{g} = -g\hat{z}$

$$\vec{g} = -\vec{\nabla}(gz)$$

equation of motion.

$$\frac{D\vec{v}}{Dt} = -\vec{\nabla}(gz) - \frac{1}{\rho}\vec{\nabla}p$$

$$= \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot(\vec{\nabla}\vec{v})$$

take dot product with $\vec{v}$

$$\vec{v}\cdot\frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v}\cdot\left[\vec{v}\cdot(\vec{\nabla}\vec{v})\right]}_{?} = -\vec{v}\cdot\vec{\nabla}(gz) - \frac{1}{\rho}\vec{v}\cdot\vec{\nabla}p$$

$$= -\vec{v}\cdot\vec{\nabla}\left(gz+\frac{p}{\rho}\right) \quad \rho \text{ const}$$

unpack ? $\quad \vec{v}\cdot\vec{\nabla} = \sum_i v_i \frac{\partial}{\partial x_i}$

$$? = \sum_j v_j\left(\sum_i\left[v_i\frac{\partial}{\partial x_i}\right]v_j\right) = \sum_i v_i \sum_j v_j \frac{\partial}{\partial x_i} v_j$$

$$= \sum_i v_i \sum_j \frac{1}{2}\frac{\partial}{\partial x_i} v_j^2 = \frac{1}{2}\vec{v}\cdot\vec{\nabla}(\vec{v}\cdot\vec{v})$$

first term $\vec{v}\cdot\frac{\partial \vec{v}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{v}\cdot\vec{v})$

so left-hand side is

$$\frac{1}{2}\left[\frac{\partial}{\partial t}v^2 + \vec{v}\cdot\vec{\nabla}v^2\right] = \frac{1}{2}\frac{Dv^2}{Dt}$$

giving

$$\frac{1}{2}\frac{Dv^2}{Dt} = -\vec{v}\cdot\vec{\nabla}\left(gz + \frac{P}{\rho}\right)$$

ρ is constant

for steady flow $\frac{\partial P}{\partial t} = 0$ so

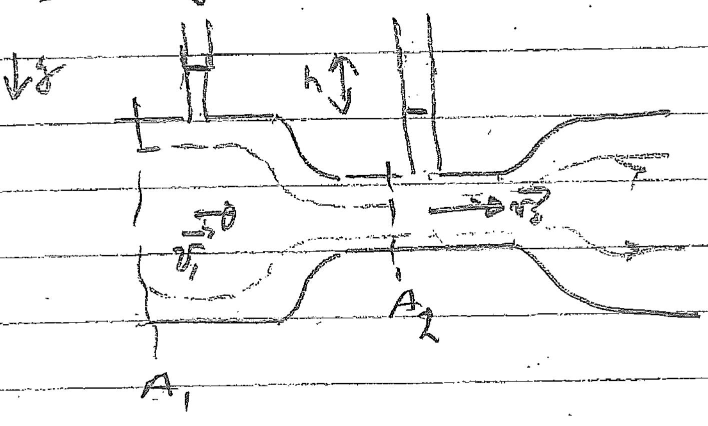
$$\vec{v}\cdot\vec{\nabla}\left(gz + \frac{P}{\rho}\right) = \frac{D}{Dt}\left(gz + \frac{P}{\rho}\right)$$

$$\frac{D}{Dt}\Big[\underbrace{\tfrac{1}{2}\rho v^2 + \rho g z + P}_{\text{energy density}}\Big] = 0$$

$\frac{F}{A} = \frac{\text{energy}}{\text{volume}}$

energy density constant as parcel moves with flow of fluid.

Venturi flow meter



continuity:

$v_1 A_1 = v_2 A_2$

$$v_2 = v_1\left(\frac{A_1}{A_2}\right) > v_1$$

Bernoulli: $\frac{1}{2}\rho v_1^2 + P_1 = \frac{1}{2}\rho v_2^2 + P_2 \Rightarrow P_1 > P_2$

$$\rho g h = P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) = \frac{1}{2}\rho v_1\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

$$v_1 = \left[\frac{gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}\right]^{1/2}$$

Waves in fluids In absence of boundary inviscid ($\eta=0$) fluid can only support longitudinal, pressure waves.

Consider pressure displacement (difference from static value)

name, not derivative

$$P(\vec{r},t) = P_0(\vec{r}) + P'(\vec{r},t)$$

similarly for density $\rho(\vec{r},t) = \rho_0(\vec{r}) + \rho'(\vec{r},t)$

equation of motion,

$$\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \vec{\nabla}P$$

equilibrium flow $\frac{D\vec{v}}{Dt} = 0 = \rho_0\vec{g} - \vec{\nabla}P_0$

time dependent (waves)

$$(\rho_0+\rho')\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right]\vec{v} = (\rho_0+\rho')\vec{g} - \vec{\nabla}(P_0+P')$$

cancel

assume ρ', P', $\vec{v}$ small so keep only first order term

$$\rho_0\frac{\partial\vec{v}}{\partial t} + \underbrace{\rho_0(\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{neglect for small } |\vec{v}|} = \rho'\vec{g} - \vec{\nabla}P'$$

linearized equation

$$\boxed{\rho_0\frac{\partial\vec{v}}{\partial t} = \rho'\vec{g} - \vec{\nabla}P'}$$

We can typically neglect gravity as well.

Recall continuity $\frac{D}{Dt}(\rho \bar{V}_p) = 0$ or

$$(\rho + \Delta\rho)(\bar{V} + \Delta\bar{V}) = \rho\bar{V}$$

$$\cancel{\rho\bar{V}} + \rho\Delta\bar{V} + \Delta\rho\bar{V} + \underbrace{\Delta\rho\Delta\bar{V}}_{\text{neglect}} = \cancel{\rho\bar{V}}$$

$$\frac{\Delta\rho}{\rho} = -\frac{\nabla V}{V} = \frac{\Delta P}{m_B}$$

for change $\Delta\rho = \rho(\vec{r},t) - \rho_0(\vec{r}) = \rho'(\vec{r},t)$

$\Delta P = P(\vec{r},t) - P_0(\vec{r}) = P'(\vec{r},t)$

$$\frac{\rho'}{\rho_0} = \frac{P'}{m_B}$$

Compare $\frac{|\rho'\vec{g}|}{\nabla P'} \approx \frac{\rho' g}{P'/\lambda}$ λ typical wavelength

$$\underset{\text{water}}{=} \frac{(10^3\,kg/m^3)(10\,m/s^2)(\sim 1\,m)}{2\,GPa} = \frac{1}{2}\times 10^{-5}$$

So we can neglect gravity term and we have

$$\boxed{\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla P'}$$

Linearized continuity

$$\frac{\partial}{\partial t}(\rho_0+\rho') + \vec{v}\cdot\vec{\nabla}(\rho_0+\rho') + (\rho_0+\rho')\,\vec{\nabla}\cdot\vec{v} = 0$$

constant in time

neglect $\rho'|\vec{v}|$ terms

now $\vec{v}\cdot\vec{\nabla}\rho_0 \ll \rho_0\,\vec{\nabla}\cdot\vec{v}$ as

$$\frac{\Delta\rho}{\rho} = \frac{\Delta P}{M_B}$$ comparing ρ, P at two nearby points in static fluid

$$\vec{\nabla}\rho_0 = \frac{\rho_0}{M_B}\vec{\nabla}P_0 = \frac{\rho_0^2}{M_B}\vec{g}$$

equilibrium condition

ratio of terms

$$\frac{|\vec{v}\cdot\vec{\nabla}\rho_0|}{|\rho_0\vec{\nabla}\cdot\vec{v}|} \cong \frac{v\rho_0^2 g/M_B}{\rho_0(v/\lambda)} = \frac{\rho_0 g\lambda}{M_B} \ll 1$$

as before

linearized continuity

$$\frac{\partial\rho'}{\partial t} = -\rho_0\vec{\nabla}\cdot\vec{v}$$

with $P' = M_B\dfrac{\rho'}{\rho}$

$$\boxed{\frac{\partial P'}{\partial t} = -M_B\,\vec{\nabla}\cdot\vec{v}}$$

Combine with eq. of motion

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p'$$

differentiate continuity

$$\frac{\partial^2 p'}{\partial t^2} = -M_B \vec{\nabla}\left(\frac{\partial \vec{v}}{\partial t}\right) = \frac{M_B}{\rho_0} \nabla^2 p'$$

pressure wave, speed

$$c_p = \sqrt{\frac{M_B}{\rho_0}} = 1.5 \text{ km/s in water}$$

<u>Wave is transverse</u>: let $p' = f(\hat{n}\cdot\vec{r} - c_p t)$

$$\frac{\partial \vec{v}}{\partial t} = \frac{1}{\rho_0}\vec{\nabla} p' = -\frac{1}{\rho_0}\vec{\nabla} f(\hat{n}\cdot\vec{r} - c_p t) = -\frac{\hat{n}}{\rho_0} f'(\hat{n}\cdot\vec{r} - c_p t)$$

$$\frac{df}{d\xi}, \quad \xi = \hat{n}\cdot\vec{r} - c_p t$$

integrating, $\vec{v} = -\frac{\hat{n}}{\rho_0}\int \frac{df}{d\xi}\, dt = \frac{\hat{n}}{c_p \rho_0} p'$

$$dt = \frac{1}{\frac{d\xi}{dt}}\, d\xi = \frac{1}{-c_p}\, d\xi$$

$\vec{v}$ in direction of propagation.

$$\frac{v}{c_p} = \frac{p'}{c_p^2 \rho_0} = \frac{p'}{M_B} = \frac{\rho'}{\rho_0} \quad \text{small}$$