Physics 304, Spring 2019

Lecture #3: Kepler Orbits

Bound $E<0$, $0 \le \varepsilon < 1$ ellipse ($\varepsilon=0$ circle)
Unbound $E \ge 0$ $\varepsilon \ge 1$ hyperbolic ($E=0$, $\varepsilon=1$ parabola)

Eccentricity for bound orbits. At turning points (r_{min}, r_{max}) $\dot{r}=0$.

Then $U_{eff}(r_{min,max}) = \frac{1}{2}\frac{\ell^2}{\mu r^2} - \frac{\gamma}{r} = E = -|E|$

$r_{min} = r(0) = \frac{c}{1+\varepsilon}$; $r_{max} = r(\pi) = \frac{c}{1-\varepsilon}$

recall $c = \ell^2/\gamma\mu$,

$$E = \frac{1}{2}\frac{\gamma c}{r_{min}^2} - \frac{\gamma}{r_{min}} = \frac{\gamma}{c}\left[\frac{1}{2}(1+\varepsilon)^2 - (1+\varepsilon)\right]$$

$r_{min} = c/1+\varepsilon$

$$E = \frac{\gamma}{c}\left[-\frac{1}{2} + \frac{\varepsilon^2}{2}\right] = \frac{\gamma}{2c}\left[\varepsilon^2 - 1\right] < 0$$

so $\varepsilon^2 = 1 + 2E\frac{c}{\gamma} = 1 + 2E\frac{\ell^2}{\gamma^2\mu}$

Eccentricity depends on energy + angular momentum

$$\varepsilon = \sqrt{1 - |E|\frac{2c}{\gamma}}$$

Circular $\varepsilon = 0$, $E = -\frac{1}{2}\frac{\gamma}{c}$

check circular orbit where $\dot{r}=0$ always, $r=c$

$$E = \frac{1}{2}\frac{\gamma c}{c^2} - \frac{\gamma}{c} = -\frac{1}{2}\frac{\gamma}{c}$$

for any ε

$$\boxed{E = -(1-\varepsilon^2)\frac{1}{2}\left(\frac{\gamma}{c}\right)}$$

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<u>Hyperbolic orbit (unbound) $E>0$, $\varepsilon>1$</u>

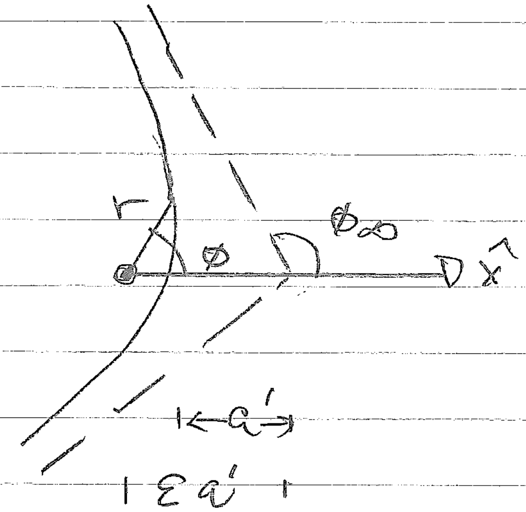
$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$ in cartesian coordinates

prob. 8.30

$$a' \equiv \frac{c}{\varepsilon^2-1} > 0 \qquad b' = \sqrt{c\,a'}$$

$$\left(\frac{x-\varepsilon a'}{a'}\right)^2 - \left(\frac{y}{b'}\right)^2 = 1$$

$$\frac{a'}{b'} = (\varepsilon^2-1)^{-1}$$



$E=0$, $\varepsilon=1$ special case. You will find parabolas

$$y^2 = c^2 - 2cx$$

We find orbits as <u>conic sections</u>.

For $\varepsilon \geq 1$ there is an asymptotic value

$r \to \infty$ as $1+\varepsilon\cos\phi \to 0$

$$\boxed{\cos\phi_\infty = -\frac{1}{\varepsilon}}$$

<u>Example</u>: Halley's comet $r_{min} = 0.59\,AU$
$\varepsilon = 0.967$

where $AU = 1.5 \times 10^8$ km is earth-sun mean distance

$$r_{max} = \frac{1+\varepsilon}{1-\varepsilon} r_{min} \approx 60\, r_{min} \approx 35\,AU >$$

orbit of Neptune

r_min = a - εa = (1-ε) a

$$\tau^2 = 4\pi^2 \left(\frac{\mu}{\gamma}\right) a^3 = \frac{4\pi^2}{G M_s} a^3$$

$\tau \cong 75\,y$ (varies due to perturbations)

Orbital Stability

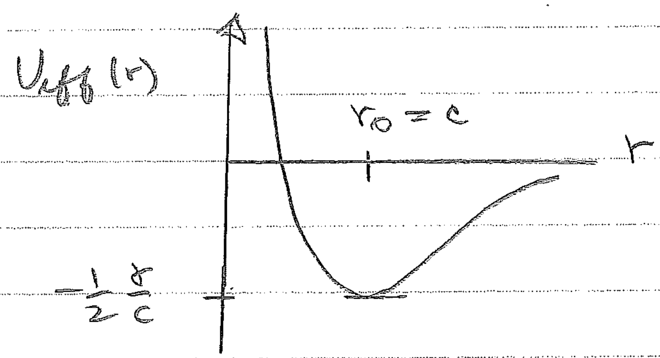
$$c = \ell^2/(\mu\gamma)$$

$$\mu\ddot{r} = -U'_{eff}(r)$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} + U(r) = \frac{\ell^2}{2\mu r^2} - \frac{\gamma}{r} = \frac{1}{2}\frac{\gamma c}{r^2} - \frac{\gamma}{r}$$

$$U'_{eff} = -\frac{\gamma c}{r^3} + \frac{\gamma}{r^2}$$

$$U'_{eff}(r_0) = 0 = \frac{-\gamma c}{r_0^3} + \frac{\gamma}{r_0^2} \Rightarrow r_0 = c$$



Circular orbit $r = r_0$ is stable if

$$U''_{eff}(r_0) > 0$$

$$U''_{eff}(r_0) = \frac{3\gamma c}{r_0^4} - \frac{2\gamma}{r_0^3} = \frac{\gamma}{c^3} \quad \text{stable}$$

More generally for $U(r)$ potential

$$U''_{eff} = \frac{3\gamma c}{r^4} + U''(r) = \frac{3\gamma c}{r^4} - F'(r)$$

with force $F(r) = -U'(r)$

$$U'(r_0) = 0 = \frac{-\gamma c}{r_0^3} - F(r_0)$$

So stability in terms of force is:

$$-\frac{3F(r_0)}{r_0} - F'(r_0) > 0$$

depends only on Force, not l. Can be shown to apply also for non-radial perturbations (changes in l).

For example $F = -\frac{k}{r^n}$ inverse power law

$$\frac{-3}{r_0}\left(-\frac{k}{r_0^n}\right) - \frac{nk}{r_0^{n+1}} > 0$$

$$(3-n) > 0$$

$$n < 3$$

$n = 3$ is special case.

Example: $F(r) = \frac{\gamma}{r^2} - \frac{\gamma'}{r^4}$

A very small γ' arises from oblateness of sun, and a small γ' from General Relativity

$$F' = \frac{2\gamma}{r^3} + \frac{4\gamma'}{r^5}$$

So condition for stability is

$$-\frac{3}{r_0}\left[\frac{-\gamma}{r_0^2} - \frac{\gamma'}{r_0^4}\right] - \frac{2\gamma}{r_0^3} - \frac{4\gamma'}{r_0^5} > 0$$

$$\frac{\gamma}{r_0^3}(3-2) + \frac{\gamma'}{r_0^5}(3-4) > 0$$

$$r_0^2\gamma - \gamma' > 0$$

$$r_0 > \sqrt{\frac{\gamma'}{\gamma}}$$

For γ' due to oblateness of the sun,

$\sqrt{\frac{\gamma'}{\gamma}} <$ solar radius.

Another example

$$F(r) = -\frac{\gamma}{r^2} \exp\left(\frac{-r}{a}\right)$$

a is finite range of gravity, due possibly from a graviton mass.

$$m_G = \frac{\hbar}{ca}$$ Here c is speed of light
$\hbar$ is planck's constant / 2π

$$F' = e^{-r/a}\left(\frac{2\gamma}{r^3} + \frac{\gamma}{ar^2}\right) = \frac{\gamma}{r^3}\left(2+\frac{r}{a}\right)e^{-r/a}$$

$$= -\frac{F}{r}\left(2+\frac{r}{a}\right)$$

stability: $$\frac{-3F}{r_0} + \frac{F}{r_0}\left(2+\frac{r_0}{a}\right) > 0$$

$$\frac{r_0}{a} < 1$$

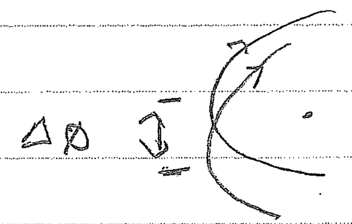
limit on graviton mass.

$M_G < 10^{-29}$ eV based on Galaxy cluster size of 58 kpc PRD 9 1119, 1974

Best limit comes from weak gravitational lensing (2004)

$$M_G < 10^{-32} \text{ eV}$$

Is elliptical orbit closed?



$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{dr/r^2}{\left[2\mu(E-U) - \frac{\ell^2}{2\mu r^2}\right]^{1/2}}$$

Closed if $\Delta\phi = 2\pi\left(\frac{a}{b}\right)$ a, b are integers.
a/b rational

You will show, radial oscillation due to perturbed orbit of potential $U(r) = kr^N$ when $kN > 0$

$$\tau_{osc} = \tau_{orbit} / \sqrt{N+2}$$

suggests closed orbit for $\sqrt{N+2}$ rational
$N = 2, -1, 7 \ldots$

Bertrand's theorem (1873) higher order perturbative analysis closed orbits stable only for $n = 2, -1$

See Am. J. Phys. Vol 56 No. 12 pp 1073-1075 (1988)

So small $\frac{\gamma'}{r^4}$ leads to precession of Mercury