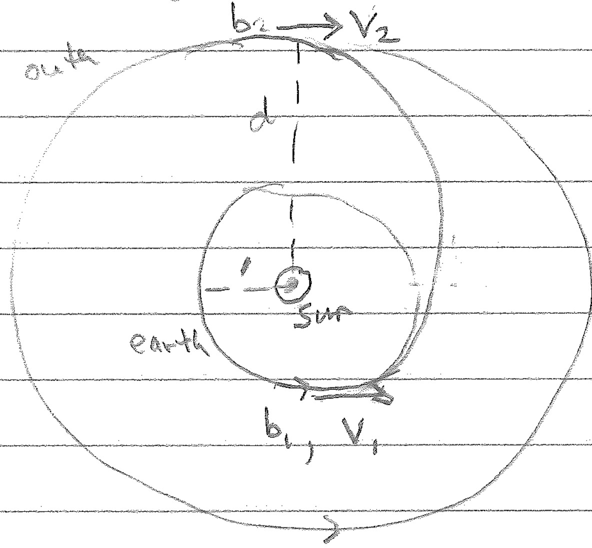
Physics 304 Spring 2018

Lecture 9: Orbital Dyamics

Hohmann transfer (1925) most energy efficient transfer between elliptical orbits. Uses two velocity changes (rocket "bursts").
We consider transfers between circular orbits

Heliocentric frame



distances in AU

transfer orbit –

$r_p = r(0) = 1$

$r_a = r(\pi) = d$

$$U_{eff} = \frac{1}{2}\frac{\gamma c}{r^2} - \frac{\gamma}{r} \qquad c = \frac{\ell^2}{\mu\gamma}$$

$\frac{c}{a} = 1 - \varepsilon^2$ c depends on orbit $\epsilon = 0,\ c = a$

$$E = -(1-\varepsilon^2)\frac{1}{2}\frac{\gamma}{c} = -\frac{1}{2}\frac{\gamma}{a}$$

$\mu \cong m_s$ mass of satellite
convenient to divide this out.

$$k \equiv \frac{\gamma}{m_s} = GM_\odot$$

$$e \equiv \frac{E}{m_s}$$

in terms of total velocity $V^2 = \dot{r}^2 + r^2\dot{\phi}^2$

$$e = \frac{1}{2}V^2 - \frac{k}{r} = -\frac{1}{2}\frac{\gamma}{a}$$

for circular orbit $(r = a)$ $\quad V = \sqrt{\frac{k}{r}}$

for earth's orbit $\quad V_e = \sqrt{\frac{k}{1AU}} = \frac{2\pi\ AU}{y} \approx 29.8\ \frac{km}{s}$

convenient to use $\boxed{\sqrt{k} = 2\pi\ \frac{AU^{3/2}}{y}}$

Time for transfer from Kepler III –

$$T_t = \frac{\tau_t}{2} = \pi\sqrt{\frac{a_t^3}{k}} \quad \text{and } a_t = \frac{1}{2}(1+d)\ AU$$

$$= \frac{1}{2}\left(1 + \frac{d}{2}\right)^{3/2} y$$

Mars $\quad d = 1.5\ AU \quad T_{mars} \approx 0.7\ y \approx 256\ d$

Uranus $\quad d = 19.2\ AU \quad T_U = 16\ y$

Gravitational Assist See Barger & Olsson Classical Mechanics
& thanks to Dan Finley

Barger & Olsson

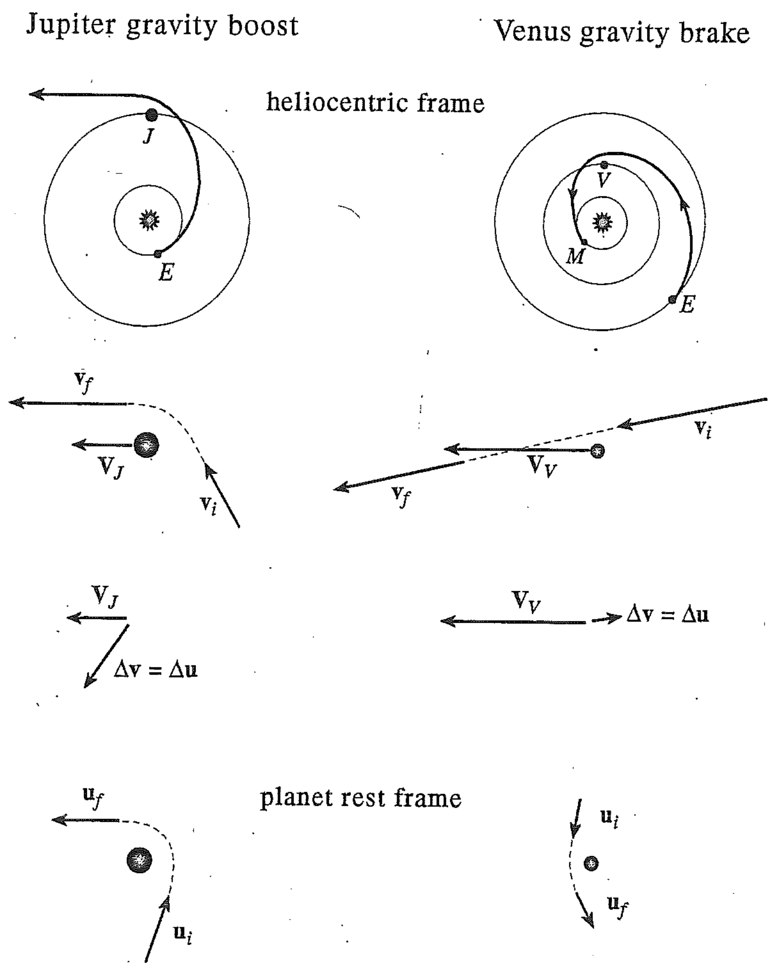


FIGURE 5-12. Velocity diagrams illustrating spacecraft velocities for a gravity boost trajectory around Jupiter and a gravity brake trajectory around Venus.

To Uranus in three Parts

Part I: elliptical orbit to Uranus (Hohmann transfer)

$r_p = r(0) = 1\ \text{AU}$

$r_a = r(\pi) = 19.2\ \text{AU}$

$a_1 = \frac{1}{2}(1 + 19.2) = 10.1\ \text{AU}$

eccentricity $\varepsilon = \dfrac{r_a - r_p}{r_a + r_p} = \dfrac{\frac{c}{1-\varepsilon_1} - \frac{c}{1+\varepsilon_1}}{\frac{c}{1-\varepsilon_1} + \frac{c}{1+\varepsilon_1}} = \varepsilon_1$

$\varepsilon_1 = \underline{0.901}$

angular momentum

$r_a r_p = \dfrac{c_1^2}{1-\varepsilon_1^2}$; $2a_1 = r_a + r_p = \dfrac{c}{1-\varepsilon_1} + \dfrac{c}{1+\varepsilon_1} = \dfrac{2c_1}{1-\varepsilon_1^2}$

$$a_1 = \frac{c_1}{1-\varepsilon^2}$$

$$c_1 = \frac{r_a r_p}{a_1}$$

$$\lambda_1 = \frac{l_1}{m} = \frac{1}{\mu}\sqrt{c_1 \mu \gamma} = \sqrt{k c_1}$$

$$= \sqrt{\frac{k r_a r_p}{a_1}} = 2\pi \left(\frac{AU^2}{y}\right) \sqrt{\frac{19.2(1)}{10.1}} = \underline{8.66 \frac{AU^2}{y}}$$

energy $e_1 = -\dfrac{k}{2a_1} = \dfrac{-1}{2(10.1)} \left(\dfrac{2\pi \, AU}{y}\right)^2$

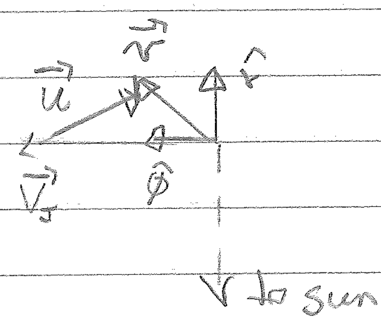
$$= -1.935 \left(\frac{AU}{y}\right)^2$$

$e < 1$ elliptical orbit

Part II: Encounter with Jupiter

near Jupiter, neglect sun's gravity. Do Galilean transformation to Jupiter rest frame.

Heliocentric frame: $\vec{v}$ satellite velocity in this frame



$\vec{u} = \vec{v} - \vec{V}_J$

incoming satellite velocity in Jupiter rest frame

$$V_J = \sqrt{\frac{k}{a_J}} = \frac{2\pi\ AU/y}{\sqrt{5.2}} = 2.76\ \frac{AU}{y}$$

Components of $\vec{v}$:

V_ϕ from $\lambda_1 = r V_\phi$ at $r = a_J$

$$V_\phi = \frac{\lambda}{a_J} = \frac{8.66}{5.2}\ \frac{AU}{y} = 1.66\ \frac{AU}{y}$$

to get v_r, use $e = \frac{1}{2}v^2 - \frac{k}{a_J} \Rightarrow v = 3.36\ AU/y$

$$V_r = \sqrt{V^2 - V_\phi^2} = 2.91\ \frac{AU}{y}$$

In Jupiter rest frame,

before collision $\vec{u} = \vec{v} - \vec{V}_J$

after collision $\vec{u}' = \vec{v}' - \vec{V}_J$

Gravitational boost $\equiv$ kinetic energy change in Heliocentric frame. ($m_s \equiv$ mass of satellite)

$$\frac{2\Delta K}{m_s} = |\vec{v}'|^2 - |\vec{v}|^2 = |\vec{u}' + \vec{V}_J|^2 - |\vec{u} + \vec{V}_J|^2$$

collision is elastic so $|\vec{u}| = |\vec{u}'| \equiv u$

$$|\vec{v}'|^2 - |\vec{v}|^2 = 2\underbrace{(\vec{u}' - \vec{u})}_{\Delta\vec{u}} \cdot \vec{V}_J = 2\Delta\vec{v} \cdot \vec{V}_J$$

$\Delta\vec{u} = \Delta\vec{v}$ is Galilean invariant

Boost $\Delta\vec{v} \cdot \vec{V}_J > 0$

Brake $\Delta\vec{v} \cdot \vec{V}_J < 0$

max ΔK, max $\Delta\vec{v} \cdot \vec{V}_J$

$\vec{v} = v_r \hat{r} + v_\phi \hat{\phi}$

$\vec{v}' = v_r' \hat{r} + v_\phi' \hat{\phi}$

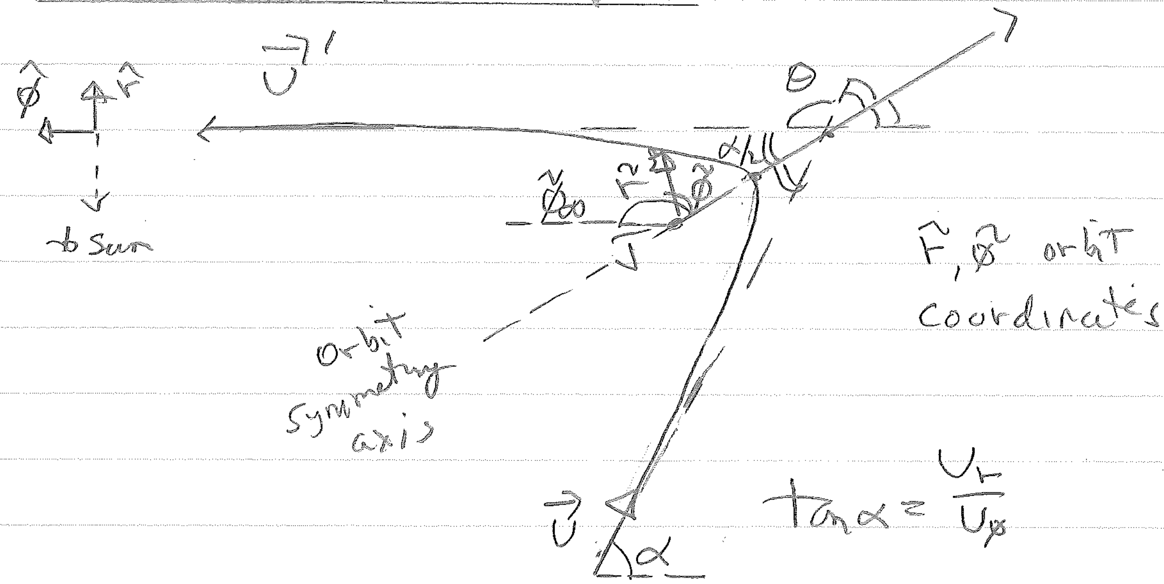
$\vec{V}_J = V_J \hat{\phi}$

$\Delta\vec{v} \cdot \vec{V}_J = (v_\phi' - v_\phi) V_J$

max when $v_r' = 0$, $\vec{v}' \parallel \vec{V}_J$ so we want

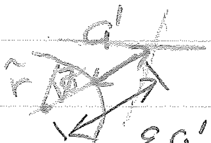
$$\boxed{\vec{v}' = (u + V_J)\,\hat{\phi}}$$

Scattering in Jupiter rest frame



$$\tan\alpha = \frac{v_r}{v_\phi}$$

$e = \frac{1}{2} v^2 > 0$ energy of Jupiter orbit



Scattering angle $\theta = \pi - \alpha$

$$r = \frac{a'(\varepsilon^2 - 1)}{1 + \varepsilon \cos\phi} \quad \text{where } a' \equiv \left(\frac{c}{\varepsilon^2 - 1}\right)$$

semi-major axis of hyperbolic orbit

$$\phi_\infty = \theta + \frac{\alpha}{2}$$

$$\tilde{r} \to \infty \text{ at } \tilde{\phi}_\infty = \cos^{-1}\left(-\frac{1}{\varepsilon}\right)$$

$$-\frac{1}{\varepsilon} = \cos\tilde{\phi}_\infty = -\cos\left(\frac{\alpha}{2}\right)$$

We get α on following page.

$\vec{u}$ components:

$$u_r = V_r = 2.91 \ \frac{AU}{y}$$

$$u_\phi = V_\phi - V_J = (1.66 - 2.76)\frac{AU}{y} = -1.09 \ \frac{AU}{y}$$

$$u = 3.11 \ \frac{AU}{y}$$

$$\tan\alpha = \frac{u_r}{u_\phi} \Rightarrow \boxed{\alpha = 70^\circ}$$

You will show that $r_p > R_J$ so orbit exists.

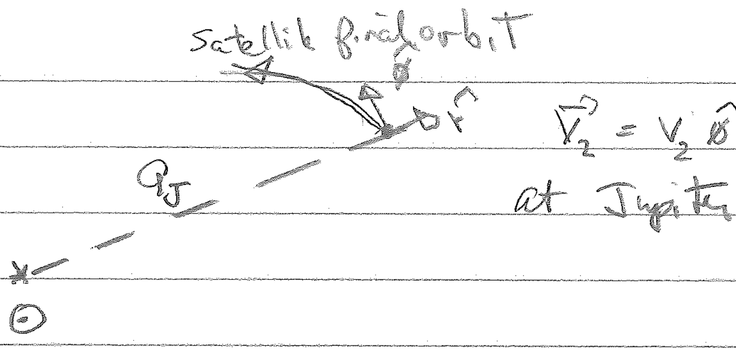
Part III Beyond Jupiter (Heliocentric frame)

$$v_2 = u' + V_J = u + V_J = (3.11 + 2.76)\frac{AU}{y} = 5.87 \ \frac{AU}{y}$$

$$\text{So } e_2 = \frac{1}{2}(v_2)^2 - \frac{k}{a_J} = \frac{1}{2}(5.87)^2 - \frac{(2\pi)^2}{5.2}$$

$$= 9.636 \left(\frac{AU}{y}\right)^2 > 0$$

final solar orbit is hyperbolic



So a_J is perihelion of final satellite orbit

Semi-major axis $a_2' = \dfrac{k}{2e_2'} = \dfrac{1}{2}\dfrac{(2\pi)^2}{9.636} = 2.05$ AU

$r_{p_2} = a_J = 5.2$ AU $= a_2'(\varepsilon_2 - 1)$ giving

$\varepsilon_2 = 3.54$ final orbit eccentricity

<u>Time to Uranus</u>: both times $E \to J$, $J - U$ are fractions of orbits so we cannot use Kepler III.

Eccentric Anomaly time dependence of $\phi(t)$

from angular momentum $\dot{\phi} = \frac{\ell}{\mu r^2}$

$$dt = \frac{\ell}{\mu} r^2 d\phi$$

in general $r = \frac{c}{1+\varepsilon \cos\phi}$

In case of circular orbit $\varepsilon = 0$, $r = a =$ constant

$$t = \frac{\ell}{\mu} a^2 \phi \qquad \text{period } T = \frac{\ell}{\mu} a^2 2\pi$$

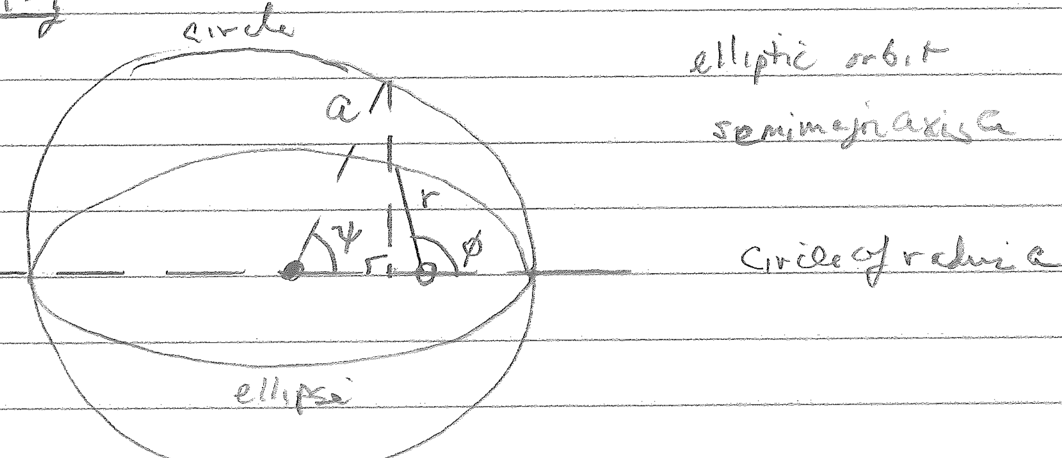
$$\phi = \frac{2\pi t}{T} \quad \text{(circular orbit only)}$$

suggest defining angle ψ

$$1 = \left(\frac{x + a\varepsilon}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \equiv \cos^2\psi + \sin^2\psi$$

$b^2 = a^2(1-\varepsilon^2)$

geometrically



you can show

$$\boxed{\tan(\phi/2) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan\frac{\psi}{2}}$$

orbit $r = a(1-\varepsilon\cos\psi)$

time $\dfrac{2\pi t}{T} = \psi - \varepsilon\sin\psi$

<u>formulas for Hyperbolic anomaly</u>.

$$\tan\left(\frac{\phi}{2}\right) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tanh\frac{\psi}{2}$$

$$r = a'(\varepsilon\cosh\psi - 1)$$

$$\sqrt{\frac{k}{a'^3}}\,t = \varepsilon\sinh\psi - \psi$$

<u>Part I</u> E-J
$$\left\{\begin{array}{l} r_J = 5.2\,AU \\ a_1 = 10.1\,AU \\ \varepsilon_1 \cong 0.9 \end{array}\right.$$
(orbit to here)

solve $r_J = a_1(1-\varepsilon_1\cos\psi_1)$ $\quad \psi_1 = 57.5^\circ$

$$\frac{2\pi t}{\underset{\text{"}32.1y}{T}} = \psi_1 - \varepsilon_1\sin\psi_1 = 0.2438$$

$$\boxed{t_1 = 1.25\,y}$$

Part II J-U hyperbolic orbit

solve $r_U = a_2(\varepsilon_2 \cosh \psi_2 - 1)$

$r_U = 19.8$ AU ; $a_2 = 2.05$; $\varepsilon_2 = 3.54$

$$\boxed{\psi_2 = 1.7367}$$

$$\underbrace{\sqrt{\frac{k}{a^3}}}_{2.14\,y^{-1}}\, t_2 = \underbrace{\varepsilon \sinh \psi_2 - \psi_2)}_{=8.00}$$

gives $t_2 = 3.74$ y

$$\boxed{\text{total time} = t_1 + t_2 = 1.25\,y + 3.74\,y = 5\,y}$$

compared to Hohmann transfer time 16 y