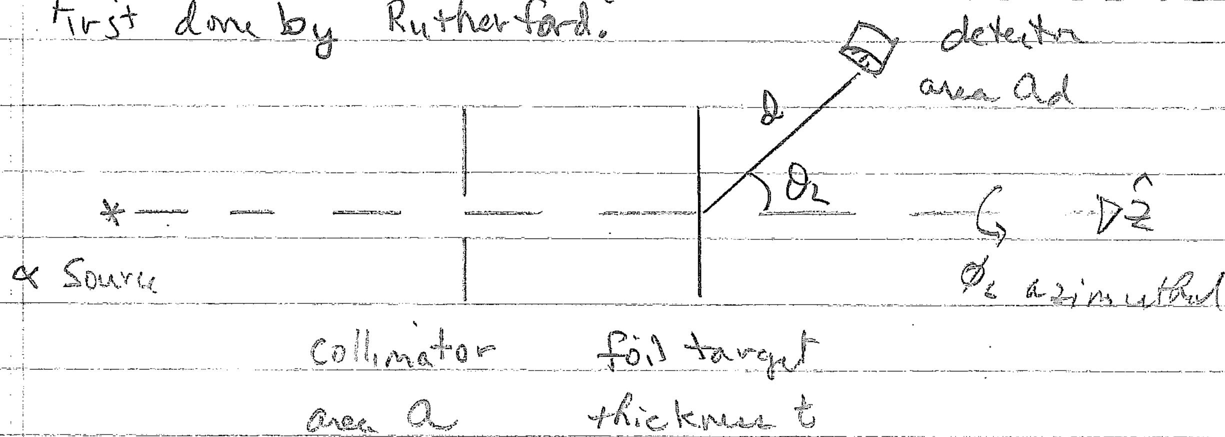
Physics 304 Spring 2019

Lec 5: Collision Theory (Ch 14)

Experiment which reveals structure of matter.
Particle beam on target ("fixed target") experiment.
First done by Rutherford.



beam flux $F \equiv \frac{\#\text{ particles}}{\text{area} * \text{time}}$

detector solid angle, area a_d, $\Delta\Omega = \frac{a_d}{d^2}$

Rate detected at angle θ_L, ϕ_L

$$\Delta R = \frac{dR}{d\Omega}(\theta_L, \phi_L)\ \Delta\Omega \qquad \frac{\#\text{detected}}{\text{time}}$$

assume azimuthal symmetry

Physical quantity of interest, differential cross section

$$\frac{d\sigma}{d\Omega}(\theta_L) = \frac{\frac{dR}{d\Omega}(\theta_L)}{F \cdot (n_t t) a}$$

$n_{sc} = \underbrace{(n_t t)}_{\text{"}n_{tar}\text{" in text}} a$ # scattering centers

n_t = number density of target

t = thickness

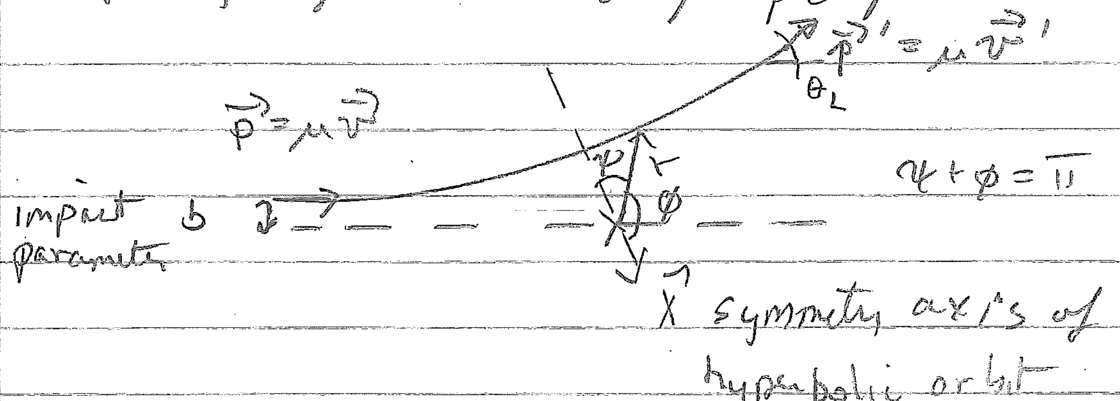
a = collimator area

Total cross section $\sigma \equiv \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$

typically measured in barns $1b = 10^{-24}\,cm^2$

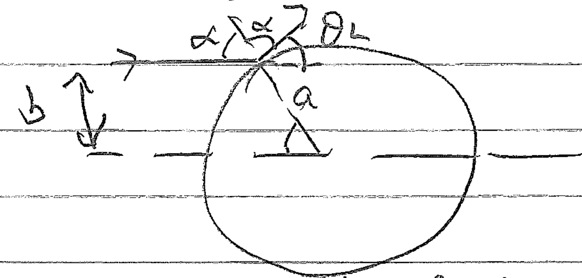
effective area $\perp$ beam. Galilean (Lorentz) invariant w.r.t. velocity transformations in beam direction

Scattering angle determined by impact parameter b



r, ϕ are here orbital parameters

Example: hard sphere scattering

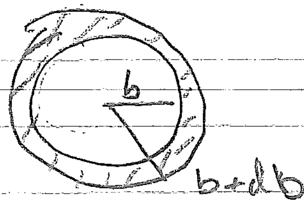


$2\alpha + \theta_L = \pi$

here α is ϕ_∞

$$\frac{b}{a} = \sin\alpha = \sin\left(\frac{\pi - \theta_L}{2}\right) = \cos\frac{\theta_L}{2}$$

particles in beam annulus $(b, b+db)$
scattered at angle θ_L between $(\theta_L, \theta_L + d\theta)$



for hard sphere, $n_{sc} = 1$

$$dR = (2\pi b)\,|db|\,F$$

$$\frac{dR}{d\Omega} = \frac{2\pi b\,F}{2\pi \sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\boxed{\frac{d\sigma}{d\Omega}(\theta_L) = \frac{b}{\sin\theta_L}\left|\frac{db}{d\theta_L}\right|}$$

$n_{sc} = 1$
azimuthal symmetry

hard sphere of radius a $\qquad b = a\cos\theta/2$

$$\frac{db}{d\theta} = -\frac{a}{2}\sin\frac{\theta}{2}$$

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{2}\,\frac{\sin\theta/2\,\cos\theta/2}{\sin\theta} = \frac{a^2}{4}$$

total cross section $\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \pi a^2$

Example 14.4 Beam of unspecified particles incident on copper foil, $t = 10\mu m = 10^{-3}$ cm

assume $\sigma = 2$ barns $= 2 \times 10^{-24}$ cm

and $N_{inc} = F(\text{area})(\text{time}) = 10^{9}$

total incident particles. Find total scattered

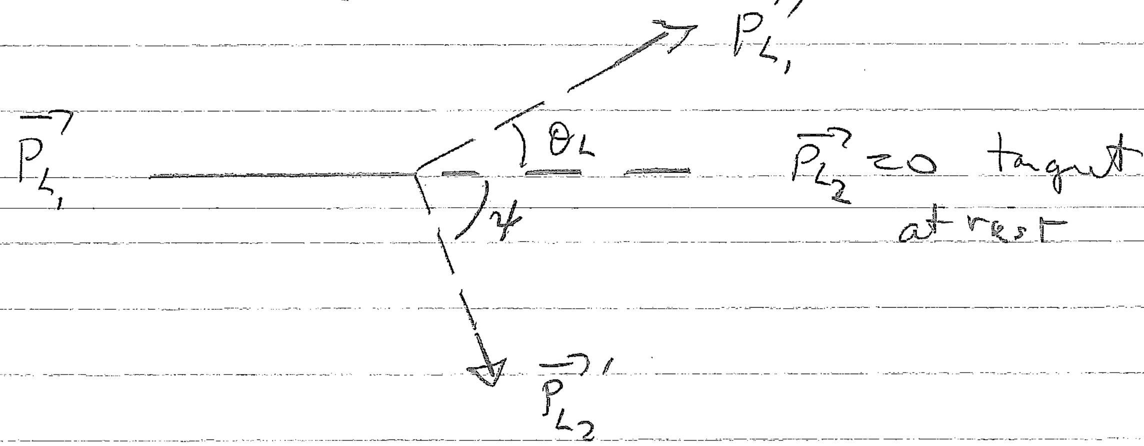
$$n = \frac{\rho}{m_{Cu}} = \frac{8.9\ g/cm^3}{(63.5\ AMU)(1.66 \times 10^{-24}\ g/AMU)}$$

$$N_{sc} = \sigma \cdot t\, n = \frac{\sigma t \rho}{m_{Cu}}$$

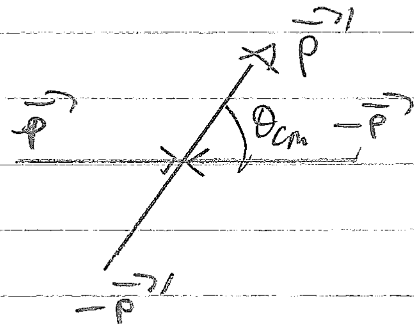
$$= \frac{(2 \times 10^{-24}\ cm^2)(10^{-3}\ cm)\ \ 8.9\ g/cm^3}{63.5\ AMU\ (1.66 \times 10^{-24}\ g/AMU)}$$

$$= \frac{2(8.9)}{63.5(1.66)} \times 10^{-3+9} = 1.69 \times 10^{5}$$

Elastic Scattering Kinematics



cm frame



elastic: $|\vec{p}'| = |\vec{p}| \equiv p$

$\vec{P}_{L_1} = m_1 \vec{V}_1$ $\qquad$ $\vec{p} = m_1 \vec{U}_1$ cm frame initial

Galilean transform with $\vec{V}_{cm} = \dfrac{m_1 \vec{V}_1}{m_1 + m_2}$

$$\vec{V}_1 = \vec{U}_1 + \vec{V}_{cm}$$

$$\vec{P}_{L_1} = m_1 \vec{V}_1 = \underbrace{m_1 \vec{U}_1}_{= \vec{p}} + \left(\frac{m_1^2}{m_1 + m_2}\right) \vec{V}_1$$

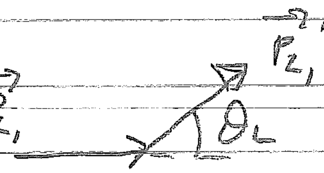
so $\quad \vec{p} = \left(1 - \dfrac{m_1}{m_1 + m_2}\right) \vec{P}_{L_1}$

$$\vec{P}_{L_1} = \left(\frac{m_1 + m_2}{m_2}\right) \vec{p} = (1 + \lambda) \vec{p} \quad \text{w/ } \lambda \equiv \frac{m_1}{m_2}$$

Similarly, $\vec{P}_{L_1}' = \lambda \vec{P} + \vec{p}'$

relation between angles

$$\tan\theta_L = \frac{\vec{P}_{L_1} \times \vec{P}_{L_1}'}{\vec{P}_{L_1} \cdot \vec{P}_{L_1}'}$$



$$= \frac{\sin\theta_{cm}}{\lambda + \cos\theta_{cm}}$$

$\boxed{\text{for } \lambda = 1,}$ $\theta_L = \frac{1}{2}\theta_{cm}$
and $\theta + \psi = \frac{\pi}{2}$

Counting rate is the same in either frame.
σ is Galilean invariant

$$\frac{d\sigma}{d\Omega_L} d\Omega_L = \frac{d\sigma}{d\Omega_{cm}} d\Omega_{cm}$$

or
$$\left(\frac{d\sigma}{d\Omega_L}\right) = \frac{d\sigma}{d\Omega_{cm}} \left|\frac{d\Omega_{cm}}{d\Omega_L}\right|$$

for azimuthal symmetry $\quad (C_{cm} \equiv \cos\theta_{cm})$

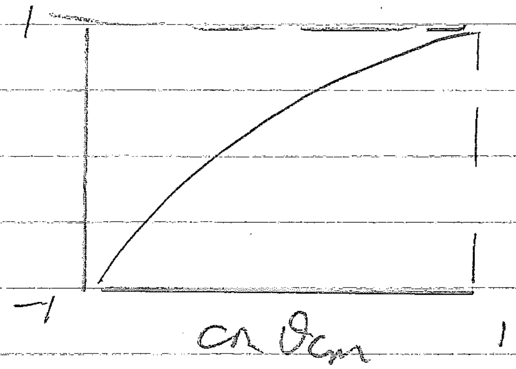
$$\left|\frac{d\Omega_{cm}}{d\Omega_L}\right| = \left|\frac{d\cos\theta_{cm}}{d\cos\theta_L}\right| = \frac{\left(1 + 2\lambda C_{cm} + \lambda^2\right)^{3/2}}{1 + \lambda C_{cm}}$$

<u>Example.</u> Take $\dfrac{d\sigma}{d\Omega_{cm}} = \dfrac{a^2}{4}$ constant, i.e. hard sphere

take $\lambda = \frac{1}{2}$

$$\frac{d\sigma}{d\Omega_L} = \frac{a^2}{4}\left(1 + \frac{\cos\theta_{cm}}{2}\right)^2$$ not constant in lab frame.

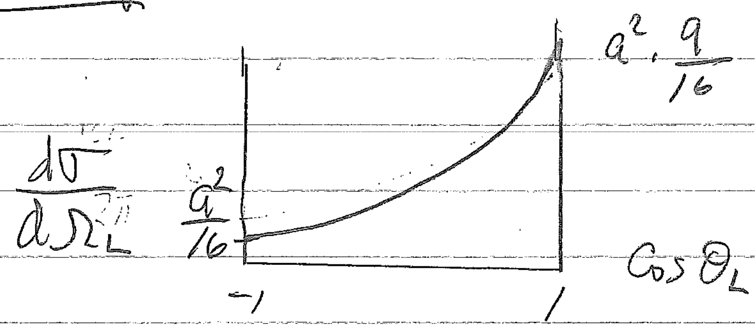
$\cos\theta_L$, $\cos\theta_{cm}$ related as (roughly



at $\cos\theta_{cm} = 1$: $\left(1 + \dfrac{\cos\theta_{cm}}{2}\right)^2 = \dfrac{9}{4}$

$\cos\theta_{cm} = -1$: $\left(1 + \dfrac{\cos\theta_{cm}}{2}\right)^2 = \dfrac{1}{4}$

<u>Sketch of</u> $d\sigma/d\Omega_L$



sharply peaked forward

$$\sigma = \int \frac{d\sigma}{d\Omega_L}\, d\Omega_L = \pi a^2$$

See link "elastic scattering kinematics"

<u>Rutherford Scattering</u> repulsive $U = \frac{+\gamma}{r}$

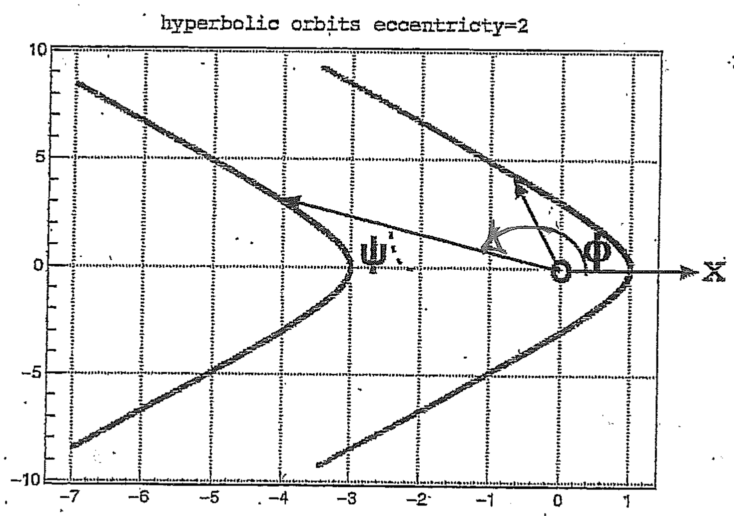
$\gamma = \alpha \hbar c Z_1 Z_2$ charges in units of e

See HW #2 where we used previous attractive solution with $\gamma \to -\gamma$, $c = \frac{\ell^2}{\mu\gamma} \to -c$.

eccentricity becomes $\varepsilon = \sqrt{1 + \frac{2Ec}{\gamma}}$ $E > 0$

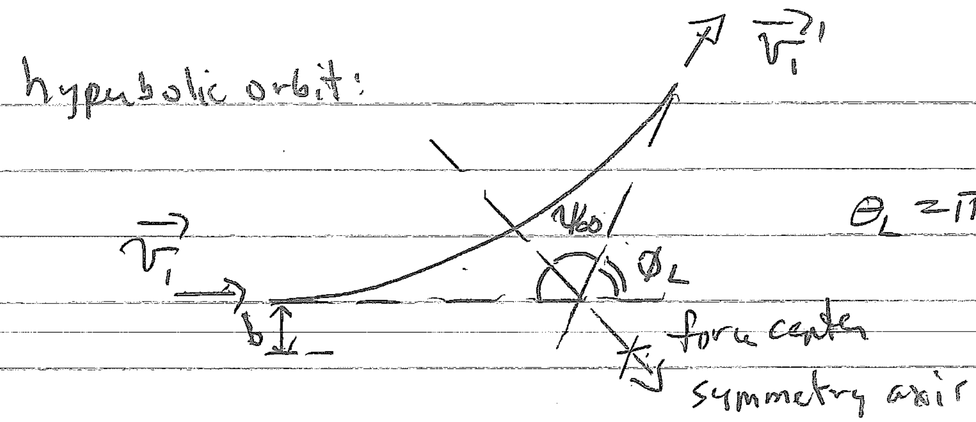
orbit equation

$$r = \frac{-c}{1+\varepsilon\cos\phi} = \frac{c}{\varepsilon\cos\psi - 1}$$



$$\psi = \pi - \phi$$

hyperbolic orbit:



$\theta_L = \pi - 2\psi_\infty$

$\varepsilon \cos\psi_\infty = 1$

$$\frac{1}{\varepsilon} = \cos\psi_\infty = \cos\left(\frac{\pi-\theta_L}{2}\right) = \sin\frac{\theta_L}{2}$$

ε depends on impact parameter b:

$E = \frac{1}{2}\mu V_1^2 \qquad \ell = \mu b V_1 = b\sqrt{2\mu E}$

$$C = \frac{\ell^2}{\mu\gamma} = \frac{2b^2E}{\gamma}$$

so $\sin\theta_L/2 = \left[1 + \frac{4b^2E^2}{\gamma^2}\right]^{-1/2}$; $\cos\theta_L/2 = \frac{2Eb}{\gamma}\left[1+\frac{4b^2E^2}{\gamma^2}\right]^{-1/2}$

$\cot\theta_L/2 = \frac{2Eb}{\gamma} \qquad \boxed{b(\theta_L) = \frac{\gamma}{2E}\cot\frac{\theta_L}{2}}$

$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}$

$$\left|\frac{db}{d\theta}\right| = \frac{\gamma}{2E}\left(\frac{1}{2}\right)\frac{1}{\sin^2\theta_L/2}$$

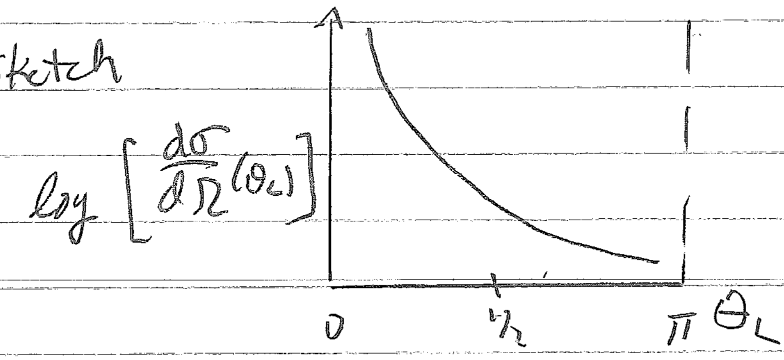
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_L}\left|\frac{db}{d\theta_L}\right| = \frac{1}{2}\frac{\gamma}{2E}\frac{1}{\sin\theta_L}\left(\frac{\cot\theta_L/2}{\sin^2\theta_L/2}\right)$$

$\sin\theta = 2\cos\theta/2 \sin\theta/2$

$$\boxed{\frac{d\sigma}{d\Omega_L} = \frac{1}{4}\left(\frac{\gamma}{2E}\right)^2\frac{1}{\sin^4\frac{\theta_L}{2}}}$$ Rutherford Scattering

<u>Note</u>: QM result is exactly the same.

sketch



example 14.6 in text

$$\frac{\frac{d\sigma}{d\Omega}(150^\circ)}{\frac{d\sigma}{d\Omega}(5^\circ)} = \frac{0.88}{2.1\times10^5} \sim 10^{-5}$$

physical size barn $b = 10^{-24}\,cm^2 = 10^{-28}\,m^2$

nucleus $\sim (fm)^2 = (10^{-15}m)^2 = 10^{-2}b$

Barn defined by $\sigma(n, U_{235})$ @ fission energy

compare to atomic scale, $(\mathring{A})^2 = (10^{-10}m)^2 = 10^8 b$

<u>Divergence at $\theta_L = 0$ is unphysical</u>,

$\theta_L \to 0$ corresponds to $b \to \infty$

but for $b \sim$ Atomic size, nuclear charge is shielded by electrons

Finite Size of Nucleus

By increasing energy of projectile, Rutherford found deviation from point nucleus.

$$r_{min} = \frac{c}{\varepsilon - 1} \qquad E = \frac{\gamma}{2c}(\varepsilon^2 - 1) \Rightarrow c = (\varepsilon^2 - 1)\frac{\gamma}{2E}$$

$$\Rightarrow \quad r_{min} = \frac{\gamma}{2E}(\varepsilon + 1)$$

$$\varepsilon^2 = 1 + \frac{4}{\gamma^2}E^2 b^2 = 1 + \cot^2\left(\frac{\theta_L}{2}\right)$$

backscatter $b = 0$, $\varepsilon = 1$, $\theta_L = \pi$

$r_{min}^{\pi} = \frac{\gamma}{E}$ | just conservation of energy |

at fixed θ_L near π, $r_{min}(E) = \frac{\gamma}{E}\left(1 + \frac{1}{4}\cot^2\frac{\theta_L}{2}\right)$

So $E \propto 1/r_{min}$ at fixed θ_L. Large θ_L probes small distances. Rutherford compared observed cross section to expected point nucleus scattering;

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{E^2}$$

as function of E at large θ_L.

α (A=4, Z=2) on Al (A=27, Z=13)

As function of energy at fixed large scattering angle

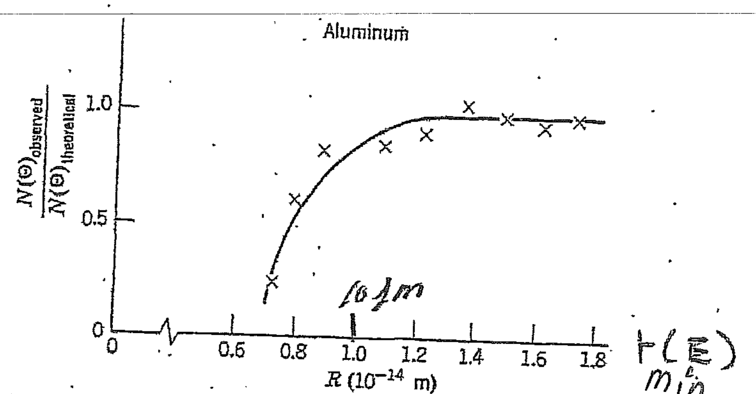


Figure 4-7 Some data obtained in the scattering of α particles from a radioactive source by aluminium. The abscissa is the distance of closest approach to the nuclear center.

From Eisberg + Resnick