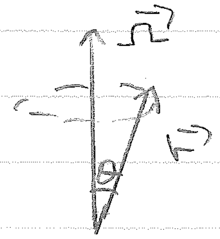
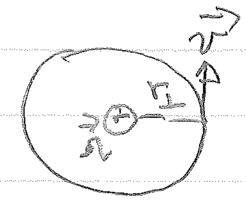
Physics 304 Spring 2019

Lec 7: Rotating Reference frames

rotated vector



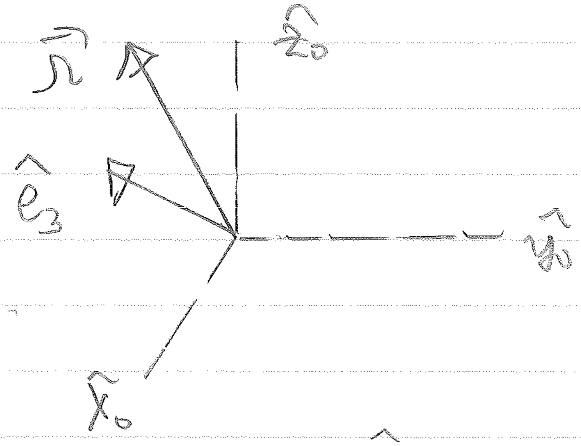
plane $\perp$ to $\vec{\Omega}$



$|\vec{v}| = r_\perp \Omega = \Omega r \sin\theta$

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{\Omega} \times \vec{r}$$

rotated reference frame $S(\hat{e}_i)$ inertial frame $S_0(\hat{x}_0, \hat{y}_0, \hat{z}_0)$



$$\frac{d\hat{e}_i}{dt} = \vec{\Omega} \times \hat{e}_i$$

Any vector $\vec{Q}$ with components Q_i in rotating frame

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_{i=1}^{3} \frac{dQ_i}{dt}\hat{e}_i + \sum_{i=1}^{3} Q_i \frac{d\hat{e}_i}{dt}$$

$$= \sum \underbrace{\left(\frac{dQ_i}{dt}\right)_S}_{\dot{Q}_i} \hat{e}_i + \sum_i Q_i (\vec{\Omega} \times \hat{e}_i)$$

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \mathring{\vec{Q}} + \vec{\Omega}\times\vec{Q}}$$

Newton's laws in rotating frame:

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \mathring{\vec{r}} + \vec{\Omega}\times\vec{r}$$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \mathring{\mathring{\vec{r}}} + \vec{\Omega}\times\mathring{\vec{r}} + \vec{\Omega}\left(\frac{d\vec{r}}{dt}\right)_{S_0}$$

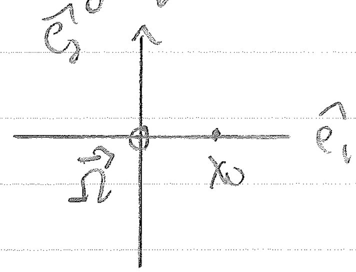
$$= \mathring{\mathring{\vec{r}}} + 2\vec{\Omega}\times\mathring{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

so $m\mathring{\mathring{\vec{r}}} = \vec{F} - 2m\,\vec{\Omega}\times\mathring{\vec{r}} - m\,\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$

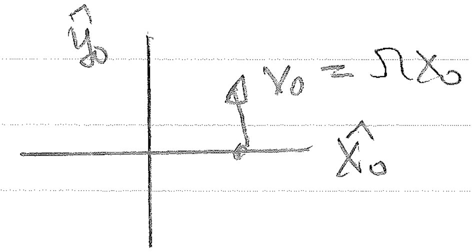
Coriolis Centrifugal

A simple example: rotating, horizontal, frictionless turntable. Mass placed at $t=0$ at x_0 at rest in rotating frame:

rotating frame $t=0$



inertial frame $t=0$



inertial frame $\vec{r}(t) = x_0\hat{x}_0 + v_0 t\,\hat{y}_0$

$$= x(t)\,\hat{e}_1 + y(t)\,\hat{e}_2$$

Components of vector transform opposite basis. Right-handed rotation of basis corresponds to left-handed rotation of coordinate.

$$\vec{r} = x(t)\,\hat{e}_1 + y(t)\,\hat{e}_2$$

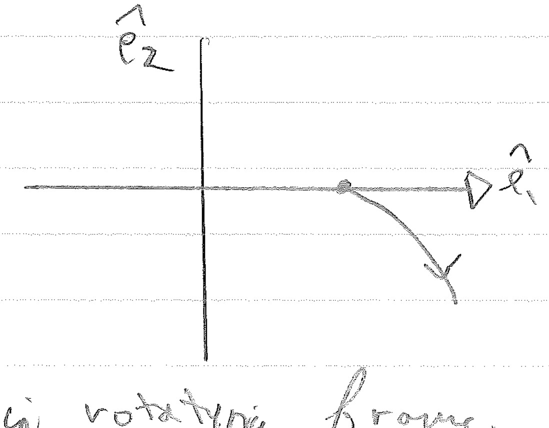
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix} \begin{pmatrix} x_0 \\ \Omega x_0 t \end{pmatrix}$$

$$x(t) = x_0 \cos\Omega t + x_0 \Omega t \sin\Omega t$$
$$y(t) = -x_0 \sin\Omega t + x_0 \Omega t \cos\Omega t$$

for small t, to order $(\Omega t)^3$

$$x(t) = x_0 + \tfrac{1}{2} x_0 (\Omega t)^2$$

$$y(t) = -x_0 \Omega t + \tfrac{1}{6} x_0 (\Omega t)^3 + x_0 \Omega t - \tfrac{1}{2} x_0 (\Omega t)^3$$
$$= -\tfrac{1}{3} x_0 (\Omega t)^3$$



path in rotating frame.

Now, use non-inertial (fictitious) forces

$$\ddot{\vec{r}} = -2\vec{\Omega}\times\dot{\vec{r}} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{\Omega}\times\dot{\vec{r}} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & 0 & \Omega \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = -\Omega\dot{y}\hat{e}_1 + \Omega\dot{x}\hat{e}_2$$

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 0 & 0 & \Omega \\ -\Omega y & \Omega x & 0 \end{vmatrix} = -\Omega^2 x\hat{e}_1 - \Omega^2 y\hat{e}_2$$
$$= -\Omega^2\vec{r}$$

$$\ddot{x} = 2\Omega\dot{y} + \Omega^2 x$$
$$\ddot{y} = -2\Omega\dot{x} + \Omega^2 y$$

let $\eta(t) = x + iy$

then $\ddot{\eta} = -2i\Omega\dot{\eta} + \Omega^2\eta \equiv RHS$

let $\eta(t) = e^{-i\Omega t}\nu(t)$ minus sign cancels term & corresponds to left-handed rotation

$$\dot{\eta} = (-i\Omega\nu + \dot{\nu})e^{-i\Omega t}$$
$$\ddot{\eta} = \left[-i\Omega\dot{\nu} + \ddot{\nu} - i\Omega(-i\Omega\nu + \dot{\nu})\right]e^{-i\Omega t}$$

$$= (-2i\Omega\dot{\nu} - \Omega^2\nu + \ddot{\nu})e^{-i\Omega t}$$

So $RHS = -2i\Omega\dot{\eta} + \Omega^2\eta = -2i\Omega(-i\Omega\nu + \dot{\nu})e^{-i\Omega t}$
$+ \Omega^2 e^{-i\Omega t}$

$$RHS = \left[-2i\,\Omega\dot{\nu} - 2\Omega^2\nu + \Omega^2\nu\right]e^{-i\Omega t} = \left(-2i\,\Omega\dot{\nu} - \Omega^2\nu\right)e^{-i\Omega t}$$

comparing to expression for $\ddot{\eta}$, $\boxed{\ddot{\nu} = 0}$

solution is $\nu(t) = \alpha + \beta t$ with α, β complex

initial condition $\eta(0) = e^{-i\Omega t}(\alpha + \beta t)\Big|_{t=0} = \alpha = X_0$

$$\dot{\eta}(0) = -\Omega\,\eta(0) + \beta = -\Omega X_0 + \beta = 0$$

$$\beta = i\,\Omega X_0$$

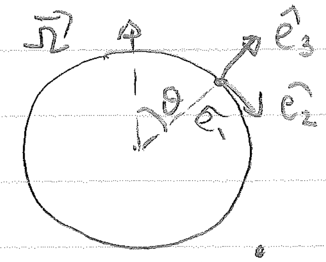
$$\eta(t) = X(t) + i\,y(t) = \left(\cos\Omega t - i\sin\Omega t\right)\left(X_0 + i\,\Omega X_0 t\right)$$

$$X(t) = X_0\cos\Omega t + X_0\Omega t\sin\Omega t$$

$$y(t) = -X_0\sin\Omega t + X_0\Omega t\cos\Omega t$$

Example ball dropped from height h (neglect air friction)
We can neglect centrifugal term,

$$\vec{F_c} = -2m\,\vec{\Omega}\times\dot{\vec{r}}$$



$\hat{e}_3$ up
$\hat{e}_2$ south
$\hat{e}_1$ west

$$\ddot{\vec{r}} = -g\,\hat{e}_3 - 2\,\vec{\Omega}\times\dot{\vec{r}}$$

to first approximation, neglect all velocity components except $\dot{z}$. then $\vec{\Omega}\times\dot{\vec{r}} = -2\Omega\sin\theta\,\dot{z}\,\hat{e}_1$

giving

$$\ddot{z} = -g$$
$$\ddot{y} = 0$$
$$\ddot{x} = 2\Omega\sin\theta\,\dot{z}$$

$$z(t) = -\tfrac{1}{2}gt^2 + h$$

$$\ddot{x} = 2\Omega\sin\theta(-gt) \Rightarrow x(t) = -\Omega\sin\theta\left(\tfrac{1}{3}gt^3\right)$$

with $t_g = \sqrt{\dfrac{2h}{g}}$

deflection $x_g = -\dfrac{\Omega}{3}\sin\theta\sqrt{\dfrac{8h^3}{g}}$ east