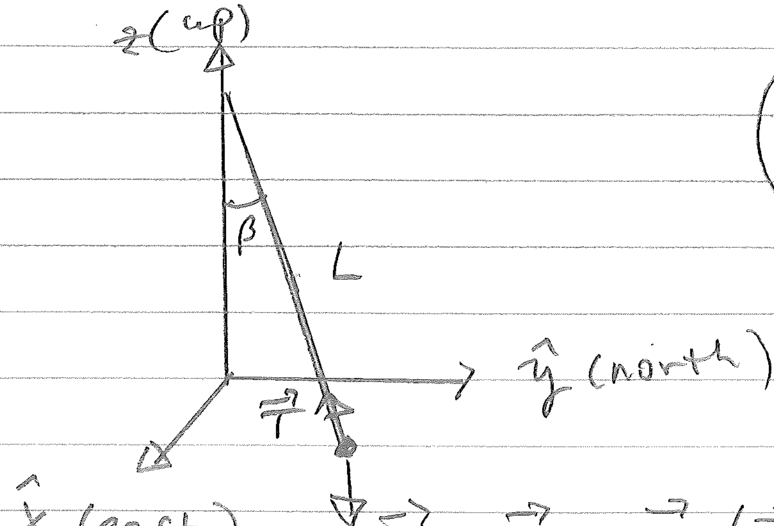
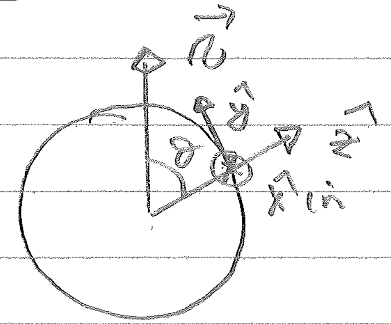
Physics 304 Spring 2015

Lecture #8: Foucault

first public exhibition of a Foucault pendulum, Paris 1851





$$\vec g = \underbrace{\vec g_0}_{\text{Newtonian gravity}} - \vec\Omega\times(\vec\Omega\times\vec r)$$ effective

$$\ddot{\vec r} = \frac{\vec T}{m} + \underbrace{g_0 - \vec\Omega\times(\vec\Omega\times\vec r)}_{\vec g} - 2\vec\Omega\times\dot{\vec r}$$

for long L, β small $\ddot{\vec r} \perp \hat z$

$$\frac{T_z}{m} = g$$ vertical components cancel

$$\frac{T_x}{T_z} = \frac{-x}{L}, \quad \frac{T_y}{T_z} = \frac{-y}{L} \qquad \begin{cases} T_x = -mg\dfrac{x}{L} \\ T_y = -mg\dfrac{y}{L} \end{cases}$$

$$\ddot{\vec r}_\perp = \frac{\vec T_\perp}{m} - 2\Omega_z\,\hat z\times\dot{\vec r}_\perp \qquad \Omega_z \equiv \Omega\cos\theta$$

Pendulum plane does not rotate in inertial (Foucault) frame.

$\vec{\omega}_F = \omega_F \hat{z}$ Vector $\vec{r}$ rotated to Foucault frame

$$\left(\frac{d\vec{r}}{dt}\right)_F = \dot{\vec{r}} + \vec{\omega}_F \times \vec{r}$$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_F = \ddot{\vec{r}} + 2\vec{\omega}_F \times \dot{\vec{r}} + \underbrace{\vec{\omega}_F \times (\vec{\omega}_F \times \vec{r})}_{\text{negligible}}$$

$$\ddot{\vec{r}}_\perp + 2\vec{\omega}_F \times \dot{\vec{r}}_\perp = \frac{\vec{T}_\perp}{m} - 2\vec{\Omega} \times \dot{\vec{r}}_\perp$$

$$\ddot{\vec{r}}_\perp = \frac{\vec{T}_\perp}{m} - 2(\Omega_z + \omega_F)\,\hat{z} \times \dot{\vec{r}}_\perp$$

$\omega_F = -\Omega_z$ will put us into Foucault frame when pendulum plane is non rotating. This is inertial frame.

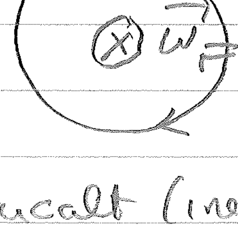
$$\ddot{x}_F = -g\frac{x_F}{L} \ ; \ \ddot{y}_F = -g\frac{y_F}{L}$$

with $x_F(0) = x_0$, $y_F(0) = y_0$ here subscript means $t=0$ and $\dot{x}_F(0) = \dot{y}_F(0) = 0$

$$x_F(t) = x_0 \cos\omega_0 t \qquad \boxed{\omega_0 \equiv \sqrt{\frac{g}{L}}}$$

$$y_F(t) = y_0 \cos\omega_0 t$$

looking down, Foucault basis rotates clockwise



$$\otimes\ \vec{\omega}_F = -\Omega_z \hat{z}$$

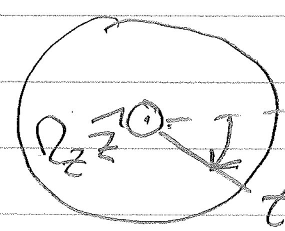
rotation to Foucault (inertial) frame

Earth basis is rotated by $\bar{R}(\Omega_z \hat{z})$ with respect to Foucault (right-handed) rotation of basis, left-handed rotation of coordinates. Components in Earth (non-inertial) frame are therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x_0 \cos\omega_0 t \\ y_0 \cos\omega_0 t \end{pmatrix}$$

$$\theta = \Omega_z t$$

Just like our first example. As seen on earth



$t = 0$

$t = \frac{2\pi}{\Omega_z}$

Foucault plane rotation

period $T_F = \frac{2\pi}{\Omega_z} = \frac{1 \text{ day}}{\cos\theta}$ (doesn't rotate at equator)

Alternatively, solving equation of motion in earth's non-inertial frame.

$$\vec{\Omega} \times \dot{\vec{r}}_\perp = \Omega_z(\hat{x}\dot{y} - \hat{y}\dot{x})$$

$$\ddot{x} = -g\frac{x}{L} + 2\Omega_z \dot{y}$$

$$\ddot{y} = -g\frac{y}{L} - 2\Omega_z \dot{x}$$

with $\eta(t) = x + iy$ becomes

$$\ddot{\eta} = -\frac{g}{L}\eta - 2i\Omega_z \dot{\eta} \qquad (*)$$

as before, let $\eta = e^{-i\Omega_z t}\nu(t)$ from differentiation

$$\ddot{\eta} = \left(-\Omega_z^2 \nu - 2i\Omega_z \dot{\nu} + \ddot{\nu}\right)e^{-i\Omega_z t}$$

$$= \left(-\frac{g}{L}\nu - 2i\Omega_z \dot{\nu}\right)e^{-i\Omega_z t} \quad \text{right hand side of } (*)$$

$$\ddot{\nu} = \left(\Omega_z^2 - \frac{g}{L}\right)\nu$$

$\omega_0 = \sqrt{\frac{g}{L}} \gg \Omega_z$ so we neglect it, giving

$$\ddot{\nu} = -\omega_0^2 \nu$$

Write $\nu(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

initial conditions $\nu(0) = x_0 + iy_0 = A + B$

$\dot{\nu}(0) = 0 = i\omega_0(A - B)$ $\Big\}$ $A = B = \frac{1}{2}(x_0 + iy_0)$

$$\eta = e^{-i\Omega_z t}\left[\frac{x_0}{2}\underbrace{\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)}_{2\cos\omega_0 t}+\frac{i y_0}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)\right]$$

$$= (\cos\Omega_z t - i\sin\Omega_z t)\cos\omega_0 t\,(x_0 + i y_0)$$

equating real & imaginary parts

$$\begin{pmatrix} x \\ y \end{pmatrix} = \cos\omega_0 t \begin{pmatrix} \cos\Omega_z t & \sin\Omega_z t \\ -\sin\Omega_z t & \cos\Omega_z t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

as before.