Physics 304 Spring 2018

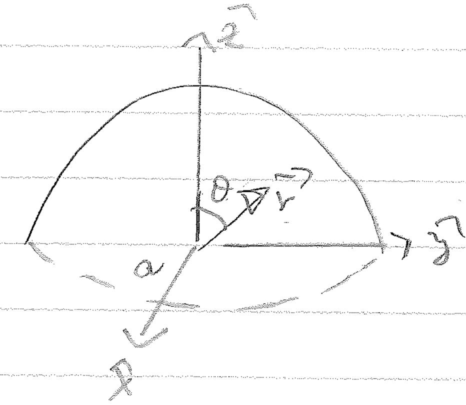
Lecture 9: Rigid Body Motion

(1) Center of mass

$$m_T = \sum_{a\ \text{particles}} m_a \Rightarrow \int \rho(\vec{r})\, d^3\vec{r} \equiv \int dm$$

$$\vec{R} = \frac{1}{m_T} \sum_a m_a \vec{r}_a \Rightarrow \frac{1}{m_T} \int \vec{r}\, dm$$

Example: uniform density hemisphere



origin at center of bottom

$$m_T = \frac{1}{4}\left(\frac{4}{3}\pi a^3 \rho\right)$$

$$\vec{R}_{cm} = \frac{\rho}{m_T} \int_0^a r^2 dr \int_0^1 dc \int_0^{2\pi} d\phi\ \vec{r}(\theta, \phi)$$

$$c \equiv \cos\theta$$

$$\vec{r} = r\cos\theta\,\hat{z} + r\sin\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$

by symmetry, only $\vec{r}\cdot\hat{z} = r\cos\theta$ contributes to vector sum

$$\vec{R}_{cm} = \hat{z}\left(\frac{2\pi\rho}{m_T}\right) \int_0^a r^3 dr \int_0^1 c\, dc = \hat{z}\left(\frac{2\pi\rho}{m_T}\right)\left(\frac{a^4}{4}\right)\frac{1}{2}$$

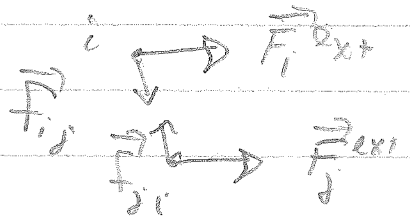
$$= \frac{3}{8} a \hat{z}$$

(2) Newton's laws for system of particles

$$\vec{P}_T = \sum_a \vec{p}_a = \underset{\text{inertial}}{\frac{d}{dt}}\left(\sum m_a \vec{r}_a\right) = m_T \frac{d}{dt}(\vec{R})$$

where $\vec{R} = \vec{R}_{cm}$ and all $\frac{d}{dt}(\;)$ refer to time derivatives w.r.t. inertial frame

Assuming internal forces are central, they cancel in equal, opposite pairs



$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$

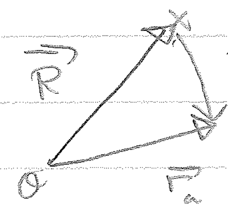
$\vec{F}_T = \sum_i F_i^{ext}$ = total external force

giving $\vec{F}_T^{ext} = \frac{d}{dt}\vec{P}_T$

In absence of external forces, $\vec{P}_T$ = const.
For convenience, choose inertial frame where $\vec{P}_T = 0$ (c.m. frame)

③ Angular momentum

$$\vec{L}_T = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a m_a \vec{r}_a \times \left(\frac{d}{dt}\vec{r}_a\right)$$



$\vec{r}_a{}' = \vec{r}_a - \vec{R}$ (relative to cm)

$$\vec{L}_T = \sum_a m_a \left(\vec{R} + \vec{r}_a{}'\right) \times \frac{d}{dt}\left(\vec{R} + \vec{r}_a{}'\right)$$

$$= \underbrace{\sum_a m_a \vec{R} \times \frac{d}{dt}\vec{R}}_{\vec{L}_{cm} \text{ @ cm}} + \underbrace{\sum_a m_a \vec{r}\,' \times \frac{d}{dt}(\vec{r}\,')}_{\vec{L}_{rel} \text{ about cm}}$$

$$+ \left(\sum m_a \vec{r}_a{}'\right) \times \frac{d}{dt}\vec{R} + \vec{R} \times \frac{d}{dt}\left(\sum_a m_a \vec{r}_a{}'\right)$$

but $\sum_a m_a \vec{r}_a{}' = \sum m_a \left(R - \vec{r}_a\right) = m_T\vec{R} - \sum m_a \vec{r}_a$
$= 0$

$\vec{L}_T = \vec{L}_{cm} + \vec{L}_{rel}$

rotational or spin

(4) Kinetic energy

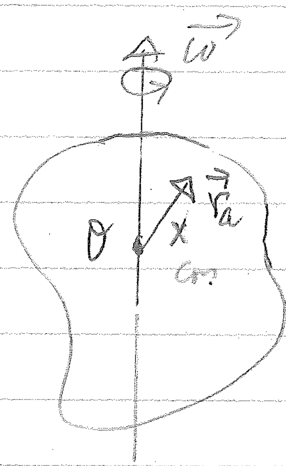
$$T = \sum_a m_a \left|\frac{d}{dt}\vec{r_a}\right|^2 = \sum_a m_a \left|\frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt}\right|^2$$

$$= \frac{1}{2}\sum m_a \left|\frac{d\vec{R}}{dt}\right|^2 + \frac{1}{2}\sum_{m_a} \left|\frac{d\vec{r}'}{dt}\right|^2$$

$$+ \frac{d\vec{R}}{dt} \cdot \frac{d}{dt}\underbrace{\left(\sum m_a \vec{r_a}'\right)}_{0} = T_{cm} + T_{rel}$$

or body

(5) Rigid Body rotated about fixed axis *



Origin O somewhere on axis of rotation.

$$\vec{V_a} = \vec{\omega} \times \vec{r_a}$$

since $\left.\frac{d\vec{r_a}}{dt}\right|_{\text{body frame}} = 0$ "rigid"

T_{body} is pure rotational

$$T_{rot} = \frac{1}{2}\sum_a m_a \left|\vec{\omega} \times \vec{r_a}\right|^2$$

* No object is perfectly rigid!

use vector identity

$$|\vec{A}\times\vec{B}|^2 = A^2B^2 - (\vec{A}\cdot\vec{B})^2$$

$$|\vec{\omega}\times\vec{r_a}|^2 = \omega^2 r_a^2 - (\vec{\omega}\cdot\vec{r_a})^2$$

$$= \sum_{i,j} \omega_i \omega_j \left(r_a^2 \delta_{ij} - r_{a_i} r_{a_j} \right)$$

$$T_{rot} = \frac{1}{2} \sum_{i,j} \omega_i \omega_j \underbrace{\left[\sum_a m_a \left(r_a^2 \delta_{ij} - r_{a_i} r_{a_j} \right) \right]}_{\equiv I_{ij}}$$

$$= \frac{1}{2} \sum \omega_i I_{ij} \omega_j = \frac{1}{2} (\omega_1, \omega_2, \omega_3) \left(I_{3\times 3} \right) \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$= \frac{1}{2} \vec{\omega}^T \cdot \underline{I} \cdot \vec{\omega}$$

will usually drop "T" for transpose as understood for left-side dot product.

$\underline{I}$ is 3x3 symmetric matrix

moment of inertia tensor.

$\bar{I}$ transforms as <u>tensor</u> under rotation group SO(3)

3x3 matrices

orthogonal

special, determinant +1

elements of SO(3) are rotation matrices

recall vector has components that transform under rotation

$$\vec{\omega}' = \bar{R}\cdot\vec{\omega}, \quad (\vec{\omega}')^T = (\vec{\omega}^T)\cdot\bar{R}^T$$

and for rotations, $\bar{R}^T = \bar{R}^{-1}$

Kinetic energy is a scalar

$$\frac{1}{2}\vec{\omega}'\cdot\bar{I}'\cdot\vec{\omega}' = \frac{1}{2}\vec{\omega}\cdot\bar{I}\cdot\vec{\omega}$$

$$= \frac{1}{2}\underbrace{(\vec{\omega}\cdot\bar{R}^{-1})}_{\vec{\omega}'}\underbrace{\bar{R}\,\bar{I}\,\bar{R}^{-1}}_{\bar{I}'}\underbrace{\bar{R}\cdot\vec{\omega}}_{\vec{\omega}'}$$

so $\boxed{\bar{I}' = \bar{R}\,\bar{I}\,\bar{R}^{-1}}$

tensor transformation

<u>elements of inertia tensor:</u>

$$I_{xx} = \sum_a m_a \left(y_a^2 + z_a^2\right) = \int dm\, \underbrace{\left(y^2+z^2\right)}_{\text{distance to x-axis}}$$

diagonal "moments of inertia"

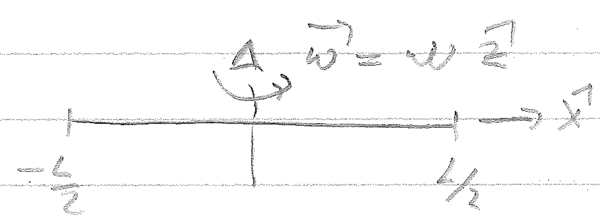
off diagonal "products of inertia"

$$I_{xy} = -\sum_a m_a x_a y_a = -\int dm\; xy$$

$$I_{xz} = -\sum m_a x_a z_a = -\int dm\; xz$$

dimension $[I_{ij}]$ = mass × (length)^2

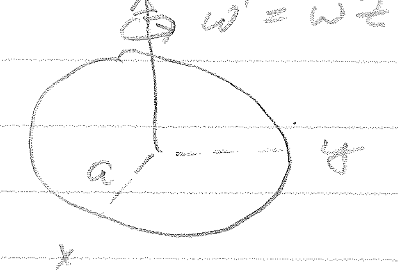
<u>Simplest example</u>: thin rod of uniform mass M length L rotated about cm



$$I_{zz} = \int dm\, (x^2 + \underbrace{y^2}_{= 0 \text{ by choice of coordinates}}) = \int_{-L/2}^{L/2} \left(\frac{M}{L}\right) dx\; x^2$$

$$= \frac{1}{3}\left(\frac{M}{L}\right) 2 \left(\frac{L}{2}\right)^3 = \frac{1}{12} M L^2$$

Example: Thin plate ("lamina")



$$\rho \Delta z = \frac{m}{\pi a^2} \equiv \sigma$$

$$dm = \sigma \rho \, d\rho \, d\phi$$

$$I_{zz} = \int_0^a dm\,(x^2+y^2) \int_0^{2\pi} d\phi = 2\pi\sigma \int_0^a \rho^3 d\rho$$

$$= 2\pi\sigma \frac{1}{4} a^4 = 2\pi \left(\frac{m}{\pi a^2}\right)\left(\frac{1}{4}\right)a^4 = \frac{1}{2} m a^2$$

Uniform sphere rotated about axis through center,

$$I_{sphere} = \frac{2}{5} m a^2$$

Hollow sphere (radius a, hole radius b)

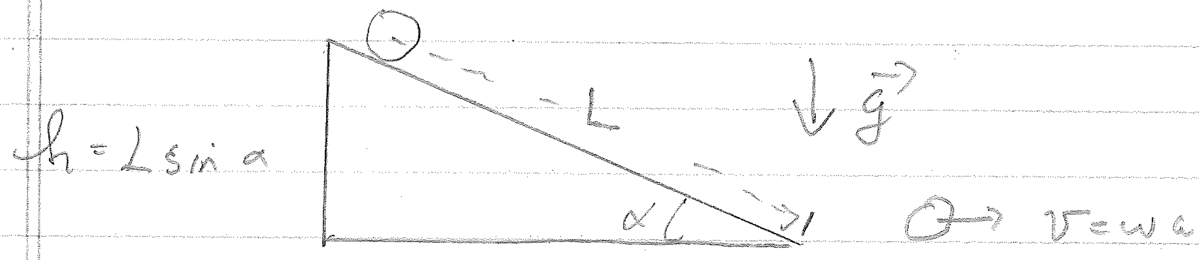
$$V' = \frac{4}{3}\pi (a^3 - b^3)$$

uniform density $\equiv \dfrac{m}{V'}$

$$I' = \frac{2}{5} m \left(\frac{a^5 - b^5}{a^3 - b^3}\right)$$

Freshman physics demo:

Which ball is solid, hollow?



cm drops distance h rolls w/o slipping

$$mgh = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mV^2\left(1 + \frac{I}{ma^2}\right)$$

$$v^2 = gh\left(\frac{ma^2}{ma^2 + I}\right)$$

$I' > I$, so solid sphere is faster.