Phys. 330 Spring 2020

HW # 2 Solution

(1) $\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} at + bx \\ dt + ex \end{pmatrix}$

$t'^2 - x'^2 = t^2 - x^2 = (at+bx)^2 - (dt+ex)^2$

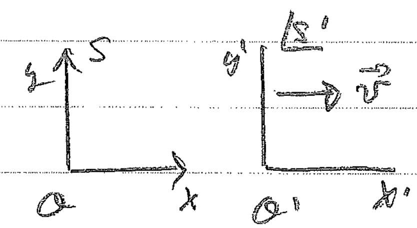
$= (a^2-d^2)t^2 - (e^2-b^2)x^2 + 2tx(ab-de)$

$ab - de = 0 \qquad \frac{d}{a} = \frac{b}{e} \equiv -\beta$ ← convention

$\left.\begin{aligned}(a^2-d^2) &= a^2(1-\beta^2) = 1 \\ (e^2-b^2) &= e^2(1-\beta^2) = 1\end{aligned}\right\} \; a^2 = e^2 = \frac{1}{1-\beta^2} = \gamma^2$

we have $\begin{pmatrix} a & b \\ d & e \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$

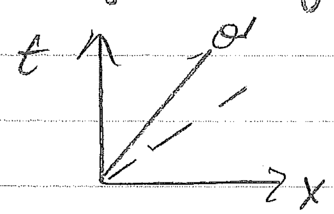
to determine β:



$\vec{v}$ in frame S

World-line of O' in frame S – 0' is x=vt in frame S

x' = 0



$\tilde{X}'_{O'}(t) = \begin{pmatrix} ct' \\ 0 \end{pmatrix}$ in S'

$\tilde{X}_{O'} = \begin{pmatrix} t \\ vt \end{pmatrix}$ in S

$\begin{pmatrix} t' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} t \\ v \end{pmatrix}$

$0 = \gamma t(1-\beta v) \Rightarrow \beta = v \;\&\; \gamma = \frac{1}{\sqrt{1-v^2}}$

Modern Physics 330: HW # 1

2

~~#~~ **Lorentz Transformation** By judiciously picking specific space-time points in a certain reference frame, show that moving clocks run slow, and that moving meter sticks are contracted. (Hints: Proper time is the time measured in the rest frame of the clock, Proper length is the length measured in the rest frame of the meter stick. A *length* in any frame is the distance between two positions measured at the same time.)

Solution:

Time dilation is shown simply by transforming from the frame in which the clock is at rest (S) to the frame in which the clock moves to the right (S') by boosting in the negative x direction from the $S \to S'$

$$\bar{X}_a \to \begin{pmatrix} c\tau \\ 0 \end{pmatrix} \to \begin{pmatrix} ct'_a \\ x'_a \end{pmatrix}.$$

We only need the time component:

$$ct'_a = \gamma(ct_a + \beta x_a) = \gamma\tau c$$

Lorentz contraction is trickier, as we must measure the stick by taking the difference of the edges at the *same time* in the frame in which the stick moves (to the right, Figure 1).

The spacetime point (event) a is:

$$\bar{X}_a \to \begin{pmatrix} 0 \\ s \end{pmatrix} \to \begin{pmatrix} ct'_a \\ x'_a \end{pmatrix}.$$

The LT gives (boosting in the negative x direction from the $S \to S'$)

$$ct'_a = \gamma(ct_a + \beta x_a) = \gamma\beta s$$
$$x'_a = \gamma(x_a + \overset{\beta ct\_a}{\cancel{\beta c t_a}}) = \gamma s$$

At time ct'_a the left edge has moved to $\beta ct'_a = \gamma\beta^2 s$. The stick length in S' is:

$$x'_a - \beta ct'_a = \gamma s - \gamma\beta^2 s = \gamma s(1-\beta^2) = s/\gamma$$

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3 **~~#~~)Simultaneity**

Given two events $\bar{X}_o \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\bar{X}_a \to \begin{pmatrix} \epsilon x \\ x \end{pmatrix}$,where $0 < \epsilon < 1$, find the boost to the frame in which these events are simultaneous.

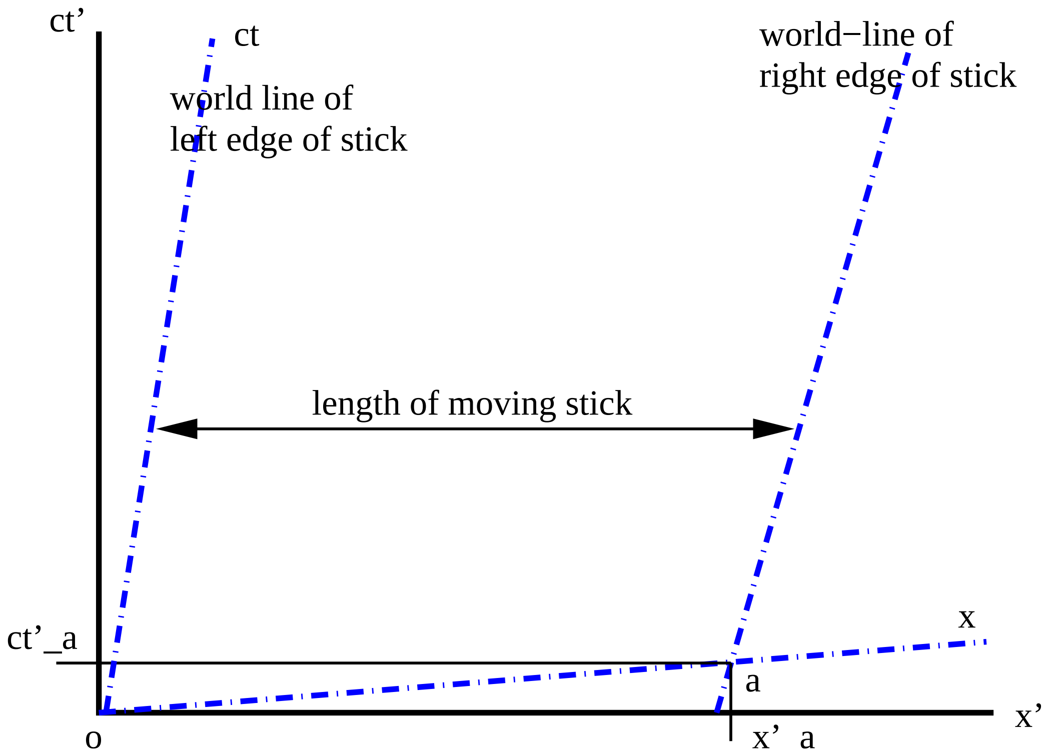

Figure 1: Measuring a moving meter stick

Solution:

We simply transform the point and solve for the value of β that gives a zero time component:

$$ct'_a = \gamma(\epsilon x - \beta x) = 0$$

So the boost is in the x-direction by $\beta = \epsilon$. Such a boost exists as long as $|\epsilon| < 1$.

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4 #~~3~~) The Train Taggers Paradox

Two graffiti "taggers" are standing next to the train tracks separated by a distance ℓ as measured in their frame (frame S). As a train traveling at near light speed goes by, they simultaneously in their frame mark the train with paint. What is the proper distance Λ between the marks, that is the distance measured by observers aboard the train (frame S$'$)?

Show that consistent answers are obtained in both frames. Place one of the taggers at the origin $x = x' = 0$ and the other at $x = \ell$. Define the following events:

1. o: common spacetime origin $\bar{X}_o \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in both frames.

2. e: $\bar{X}_e \to \begin{pmatrix} 0 \\ \ell \end{pmatrix}$ in frame S.

3. f: $\bar{X}_f \to \begin{pmatrix} 0 \\ \Lambda \end{pmatrix}$ in frame S′.

Determine the coordinates for these events in both frames. Describe how these events are observed in each frame.

Solution:
Tagger Frame S: Let's take the train frame S' to be moving in the negative x direction in the tagger frame S. The taggers observe a Lorentz contracted train, and therefore they conclude that the marks they make on the train are a factor γ further apart then they are. They conclude the proper length between the marks is $\Lambda = \gamma\ell$.

Train Frame S′: Because we are measuring the proper distance and the marks are at rest, we don't have to worry about measuring the positions at equal times. Therefore, we can simply transform the event coordinates X_e:

$$\begin{pmatrix} ct'_e \\ \Lambda \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \ell \end{pmatrix} = \begin{pmatrix} +\gamma\beta\ell \\ \gamma\ell \end{pmatrix}$$

Thus, $\Lambda = \gamma\ell$.

Interpretation: In the train frame S′, the mark e is made at time $ct'_e = \gamma\beta\ell$ after the mark at o. In this frame the taggers are separated by a distance ℓ/γ. However, between the time the marks are made, the taggers have moved in the $+x'$ direction a distance $\beta ct'_e = \gamma\beta^2\ell$. Thus we get for the total distance between the marks,

$$\Lambda = \frac{\ell}{\gamma} + \gamma\beta^2\ell = \gamma\ell(\gamma^{-2} + \beta^2) = \gamma\ell$$

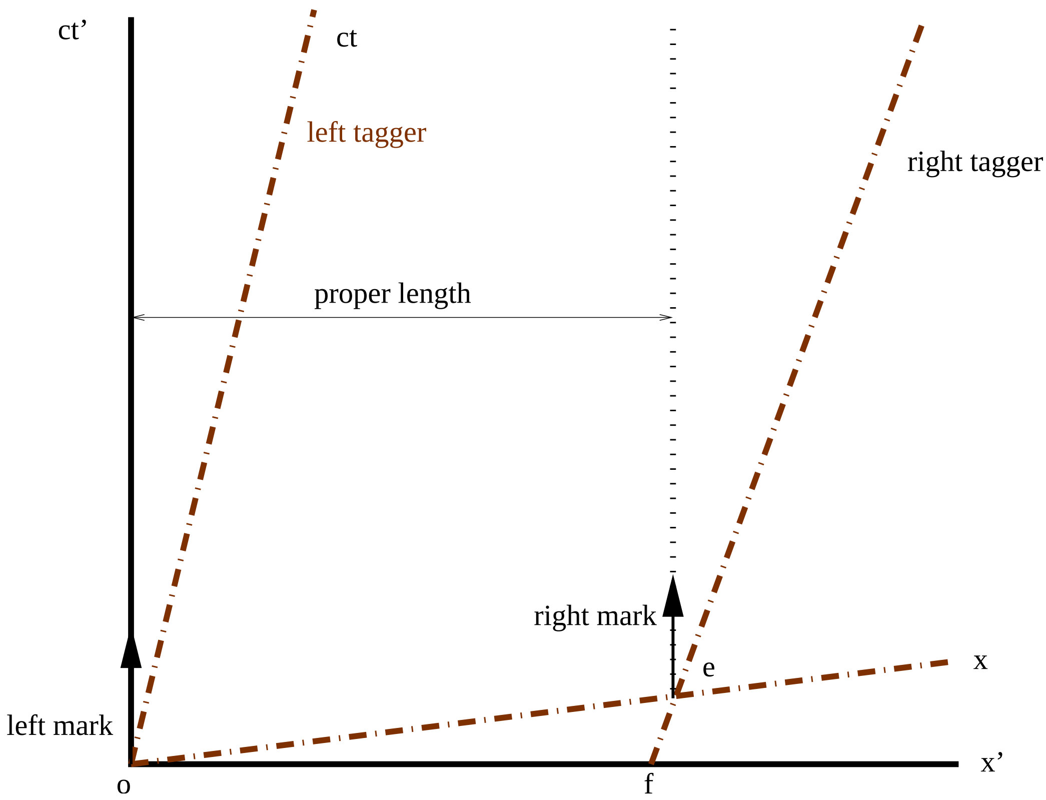
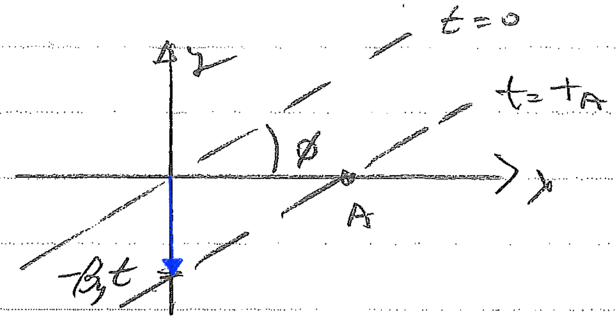


Figure 2: Events as seen in the train frame. The taggers move to the right with speed v along the dot-dashed world-lines, separated by a distance $x'_f - x'_o = \ell/\gamma$. The left mark is made at point o but the right mark is made later at point time t'_e, having moved from x'_f to x'_e. The marks proceed on vertical world-lines in this frame (they do not move).

Hw #2-7

scissors paradox: scissors move down

#28 a)



$$x_A = \frac{\beta_y t}{\tan\phi}$$

$$\dot{x}_A = \frac{\beta_y}{\tan\phi}$$

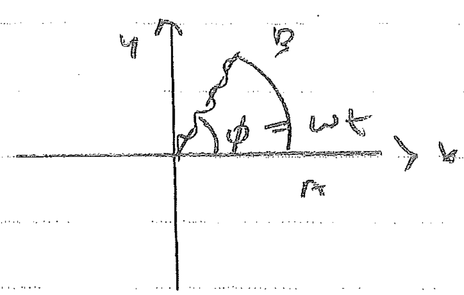
$\frac{\beta_y}{\tan\phi} > 1$ is possible

(b) Observer at x_A seems to be able to infer time that rod is struck as

$$t = \frac{\tan\phi \, x_A}{\beta_y}$$

However, this assumes rod is infinitely rigid which it cannot be.

(c)



rotating searchlight beam warns B of bullet shot at A

apperent speed $= r\omega > c$

$r > \omega/c$

No warning has been transmitted as B ~~must know~~ in advance that A will shoot.

(d) phase velocities can travel $> c$ but they do not transmit information

(38) L.T. is $\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \beta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$

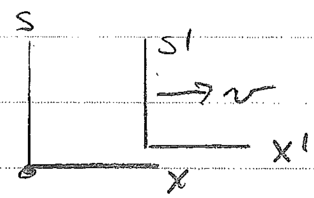
in terms of boost angle θ

$$\left.\begin{aligned} \beta &= \tanh\theta \\ \gamma\beta &= \sinh\theta \approx \theta \\ \gamma &= \cosh\theta \approx 1 \end{aligned}\right\} \beta \ll 1$$

then L.T. is $x' = x - \beta t$

$t' = t - \beta x$

Galilean transformation



origin of s moves to the left in S' frame

$$X' = x - v\, t_{sec}$$
$$t'_{sec} = t_{sec}$$

But in L.T., $X' = x - \beta c\, t_{sec} = x - v\, t_{sec}$

put back speed of light c

$$t'_{sec} = \frac{t - \beta x}{c} = t_{sec} - \frac{\beta}{c} x$$

$$= t_{sec} - \frac{v}{c^2} x \;\hat{=}\; t_{sec} \text{ to this order.}$$

(40) Collision of equal mass particles

Lab $\longrightarrow \circ \rightarrow \hat{x}$ $\qquad \tilde{P}_{TOT} = m\begin{pmatrix}\gamma+1\\ \gamma v\end{pmatrix}_{Lab}$

$P_1 = m\gamma v \qquad P_2 = 0$

$E_1 = m\gamma \qquad E_2 = m$

Cm frame $\tilde{P}_{TOT} = \begin{pmatrix}E_{cm}\\ 0\end{pmatrix}$

$$\begin{pmatrix}E_m\\ 0\end{pmatrix} = \gamma_r\begin{pmatrix}1 & -\beta_r\\ \beta_r & 1\end{pmatrix} m \begin{pmatrix}\gamma+1\\ \gamma v\end{pmatrix}$$

$$0 = m\gamma_r\left((\gamma+1)\beta_r + \gamma v\right)$$

$$\beta_r = \frac{\gamma v}{\gamma+1}$$

note $\quad \tanh\theta_r = \dfrac{\sinh\theta}{\cosh\theta+1} = \tanh\dfrac{\theta}{2}$

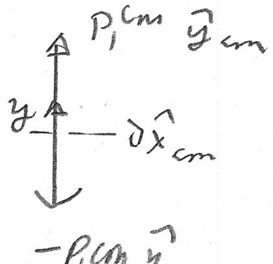
$$p_1^{cm} = \gamma_r(P_1 - \beta_r E_1) = m\gamma_r(\gamma v - \beta_r\gamma)$$

$$= m\gamma_r\gamma\left(v - \frac{\gamma v}{\gamma+1}\right) = m\gamma_r\gamma\left(\frac{v(\gamma+1)-\gamma v}{\gamma+1}\right)$$

$$= m\gamma_r\gamma\left(\frac{v}{\gamma+1}\right) = m\gamma_r\beta_r$$

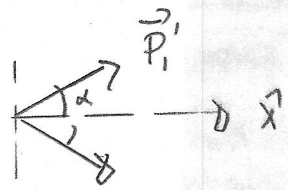
$$p_2^{cm} = \gamma_r(P_2 - \beta_r E_2) = -m\gamma_r\beta_r$$

after collision,



HW #2 - 10

Transform back to lab



$$P_{1x}' = \gamma_r (0 + m\gamma_r\beta_r) = m\beta_r\gamma_r^2$$

$$P_{1y}' = P_{1y}^{cm} = m\gamma_r\beta_r$$

$$\tan\alpha = \frac{P_{1y}'}{P_{1x}'} = \frac{1}{\gamma_r} = \sqrt{1-\beta_r^2} = \sqrt{1-\frac{v^2}{(\gamma+1)^2}}$$

$$\underset{v<<1}{\approx} 1 - \frac{1}{2}\frac{v^2}{4}$$

expand $\tan\alpha = \tan\left(\frac{\pi}{4} - \frac{\varepsilon}{2}\right) \approx 1 - \varepsilon$

find $\frac{v^2}{8} = \varepsilon$ for $\varepsilon = \frac{1}{100}$

$$v^2 = \frac{4}{50} \approx \left(\frac{2}{7}\right)^2$$

$$v = 2/7$$