Modern Physics 330: HW # 2

#1)The Lorentz Transformation

Show that the most general linear transformation taking coordinate system S to coordinate system S′, where S′ moves along the common $+x$ direction with speed v in frame S, that leaves the interval invariant is the Lorentz transformation. The invariant interval is

$$(c\Delta t)^2 - (x)^2 = (c\Delta t')^2 - (x')^2$$

#2) Time Dilation and Length Contraction

By judiciously picking specific space-time points in a certain reference frame, show that moving clocks run slow, and that moving meter sticks are contracted. (Hints: Proper time is the time measured in the rest frame of the clock, Proper length is the length measured in the rest frame of the meter stick. A *length* in any frame is the distance between two positions measured at the same time.) Illustrate the measurement of the length of the stick in the moving frame with a space-time diagram.

#3)Simultaneity

Given two events $\bar{X}_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\bar{X}_a = \begin{pmatrix} \epsilon x \\ x \end{pmatrix}$,where $0 < \epsilon < 1$, find the boost to the frame in which these events are simultaneous.

#4)The Train Taggers Paradox

Two graffiti “taggers” are standing next to the train tracks separated by a distance ℓ as measured in their frame (frame S). As a train traveling at near light speed goes by, they simultaneously in their frame mark the train with paint. What is the proper distance Λ between the marks, that is the distance measured by observers aboard the train (frame S′)?

Show that consistent answers are obtained in both frames.

Place one of the taggers at the origin $x = x' = 0$ and the other at $x = \ell$. Define the following events:

1. o: common space-time origin $\bar{X}_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in both frames.

2. e: $\bar{X}_e = \begin{pmatrix} 0 \\ \ell \end{pmatrix}$ in frame S.

3. f: $\bar{X}_f = \begin{pmatrix} 0 \\ \Lambda \end{pmatrix}'$ in frame S′.

Determine the coordinates for these events in both frames. Describe how these events are observed in each frame. Draw a space-time diagram showing events o,e,f in the train frame.

28.* Things that move faster than light‡

The Lorentz transformation equations have no meaning if the relative velocity of the two frames is greater than the velocity of light. This is taken to imply that mass, energy, and information (messages) cannot be moved from place to place faster than the speed of light. Check this implication in the following examples.

(a) The scissors paradox. A very long straight rod, which is inclined at an angle ϕ with the x axis, moves downward with uniform speed β^y (Fig. 44). Find the speed β_A of the point of intersection A of the lower edge of the stick with the x axis. Can this speed be greater than the speed of light? Can it be used to transmit a *message* from the origin to someone far out on the x axis?

(b) Suppose that the same rod is initially at rest with the point of intersection A at the origin. The region of the rod which is centered on the origin is struck by the downward blow of a hammer. The point of intersection moves to the right. Can this motion of the point of intersection be used to transmit a message faster than the speed of light?

†For reprints of several articles on the clock paradox, together with references to many more articles, see *Special Relativity Theory*, Selected Reprints, published for the American Association of Physics Teachers by the American Institute of Physics, 335 East 45th Street, New York 17, New York, 1963.

‡See Milton A. Rothman, "Things that go Faster than Light," Scientific American **203**, 142 (July, 1960).

HW # 2

Phys 330

Spring 2020

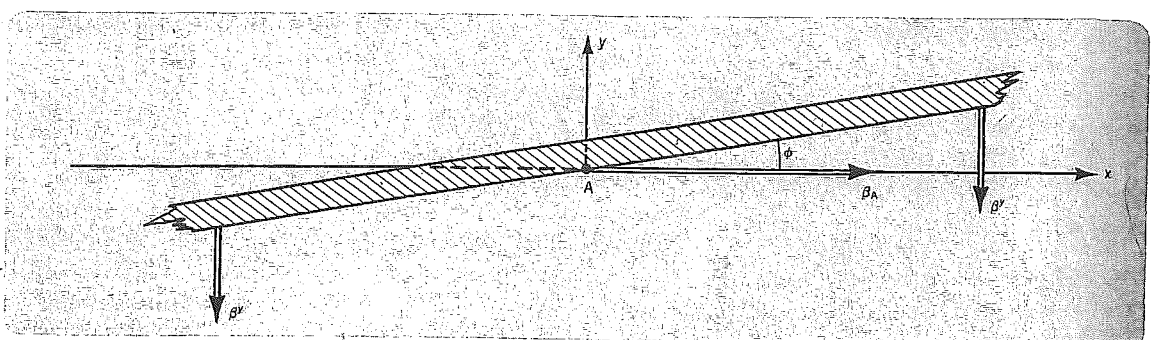


Fig. 44. Can the point of intersection A move with a speed greater than the speed of light?

(c) A very powerful searchlight is rotated rapidly in such a way that its beam sweeps out a flat plane. Observers A and B are on the plane and each the same distance from the searchlight but not near to each other. How far from the searchlight must A and B be in order that the searchlight beam will sweep from A to B faster than a light signal could travel from A to B? Before they took their positions, the two observers were given the following instructions:

To A: "When you see the searchlight beam, fire a bullet at B."

To B: "When you see the searchlight beam, duck because A has fired a bullet at you."

Under these circumstances, has not a warning gone from A to B with a speed faster than that of light?

(d) The manufacturers of some oscilloscopes claim writing speeds in excess of the speed of light. Is this possible?

Therefore the Euclidean transformation equations (inverse of Eqs. 29) become

$$\begin{aligned} x' &= x \cos\theta_r - y \sin\theta_r \approx x - \theta_r y \\ y' &= x \sin\theta_r + y \cos\theta_r \approx \theta_r x + y \end{aligned} \tag{56}$$

This approximate transformation can be made as accurate as desired by making θ_r sufficiently small.

38. The Galilean transformation

Suppose that β_r is very small. Then $\beta_r = \tanh\theta_r \approx \theta_r$. Use the series expansions of Table 8 to show that if terms that contain powers of θ_r higher than the first are neglected, the transformation equations become

$$x' = x - \beta_r t \tag{57}$$

$$(\beta_r \ll 1)$$

$$t' = -\beta_r x + t \tag{58}$$

Now use everyday, nonrelativistic Newtonian arguments to derive the transformation equations between two reference frames. These are called the *Galilean transformation equations*

$$x' = x - v_r t_{sec} \tag{59}$$

(Galilean transformation)

$$t'_{sec} = t_{sec} \tag{60}$$

where v_r is the relative speed between the two frames in meters per second.

Transformation equations 57 and 58 appear to be completely inconsistent with Eqs. 59 and 60. Is this first impression *correct*, and if not, why not? (Discussion: Why does v_r in the Galilean transformation (Eq. 59) replace β_r in Eq. 57? How does Eq. 58 look when rewritten in terms of v_r and t_{sec}? How do everyday velocities compare with the speed of light?)

39.* Limits of accuracy of a Galilean transformation

Make a more accurate approximation of the transformation equations at low relative velocities by allowing terms in θ_r^2 to remain but, again, neglecting terms with higher powers of θ_r. (This is called a *second order approximation* in θ_r. Notice from the series expansion of $\tanh\theta$ in Table 8 that even to second order in θ_r, $\beta_r \approx \theta_r$.) Show that the coefficients for x and t in Eqs. 57 and 58 agree with the improved second-order approximation to better than 1 percent for velocities β_r less than 1/7.

If a sports car can accelerate uniformly from rest to 60 miles per hour (about 27 meters per second) in 7 seconds, roughly how many days would it take to reach $\beta = 1/7$ at the same constant acceleration? How many days would be required to reach this speed at the greatest acceleration that the human body can stand for reasonable periods (about 7 g, or 7 times the acceleration of gravity)?

40.* Collisions Newtonian and relativistic —and the domain within which the two predictions agree to one percent

Proton A collides elastically with proton B, which is initially at rest. The outcome of the collision cannot be predicted. It depends upon the closeness of the encounter. In most events proton A will be deviated by only a slight angle α_A from its original direction of motion. Then proton B will be given only a slight kick off to the side, at an angle α_B (relative to the forward direction) that is close to 90 degrees. Occasionally there is a very close encounter in which B acquires nearly all the energy and goes off at a very small angle α_B to the forward direction. Between these two ex-

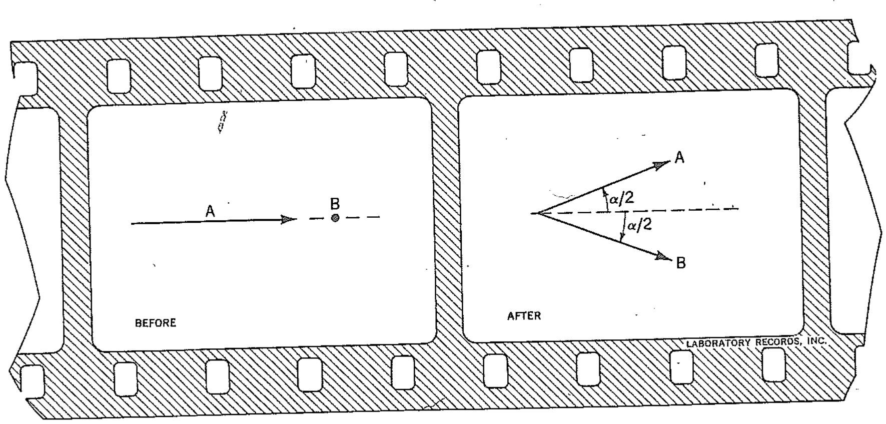
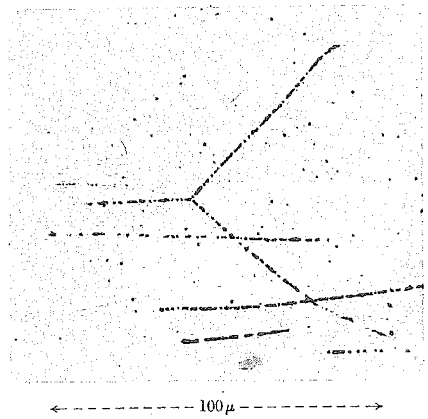


Fig. 53. Laboratory frame record of a symmetric elastic collision.

Fig. 54, A. Photograph of a nonrelativistic symmetric elastic collision between a moving proton and a second proton initially at rest. Initial speed of the incident proton is about $\beta = 0.1$. The angle between outgoing protons is 90 degrees, as predicted by Newtonian mechanics. From C. F. Powell and G. P. S. Occhialini, *Nuclear Physics in Photographs* (Oxford University Press, Oxford, 1947).



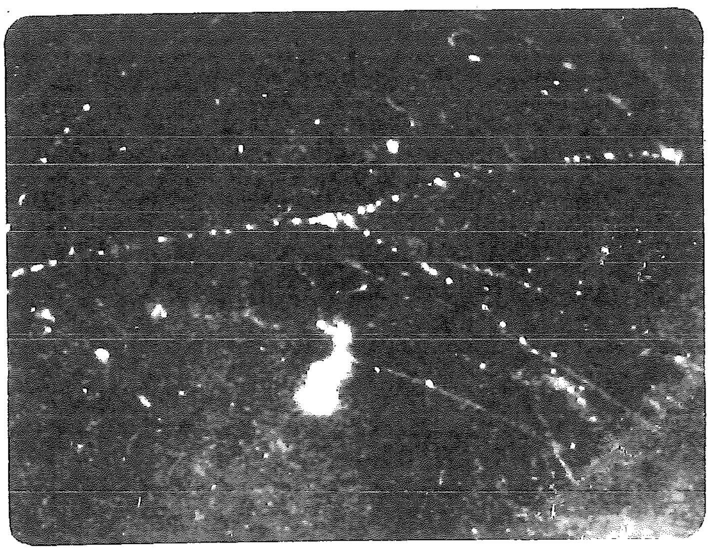


Fig. 54, B. Expansion chamber photograph of a relativistic and approximately symmetric elastic collision between a moving electron and a second electron initially at rest. Initial speed of the incident electron is about $\beta = 0.97$. The angle between outgoing electrons is much less than the 90 degree angle predicted by Newtonian mechanics. The curved path of the charged electrons is due to the presence of a magnetic field used to determine the momentum of each electron. Document Hermann Publishers, Paris.

tremes there occurs from time to time a "symmetric collision" in which the two identical particles come off with identical speeds along paths that make identical angles $\alpha_A = \alpha_B = \alpha/2$ with the forward direction (Fig. 53). *Question: How great is the angle of deflection in a symmetric collision? Discussion:* According to *Newtonian mechanics* the total angle of separation, is 90 degrees in every elastic collision (symmetric or not!). *That this angle will be less than 90 degrees for a fast impact is one of the most interesting and decisive predictions of relativity.* Figure 54,A, shows a low-velocity collision whose 90 degree angle of separation satisfies the Newtonian prediction. In contrast, Fig. 54,B, shows a high-velocity collision whose angle of separation is decisively less than 90 degrees. This circumstance means that *the difference between the separation angle from 90 degrees provides a useful measure of the departure from Newtonian mechanics.* For example, ask this question: How high must the velocity in such collision experiments be before the separation angle deviates from 90 degrees by as much as 1/100 of a radian? It greatly simplifies the analysis of this question to look at the symmetric collision pictured above from a frame of reference so chosen that one can capitalize on *symmetry arguments.* For this purpose climb onto a rocket and travel to the right with a velocity just great enough to keep up with the forward velocities of A and B after the collision. Viewed from this rocket, particles A and B therefore have *no forward velocity component:*

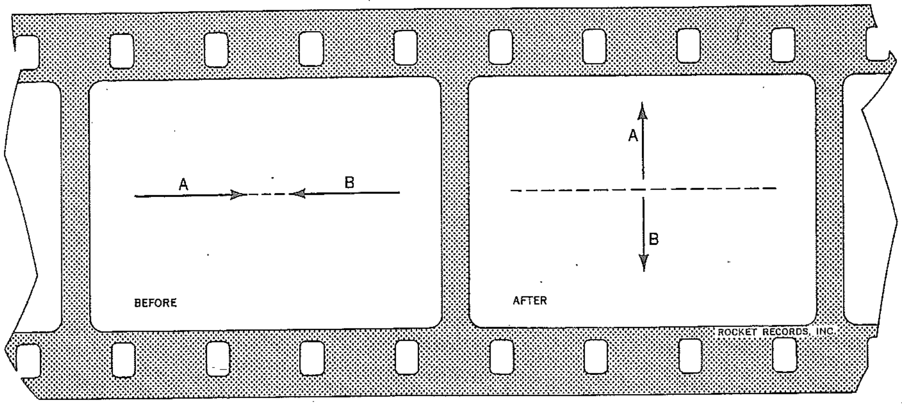


Fig. 55. Rocket frame records of the symmetric elastic collision of Fig. 53. Rocket frame is so chosen that particles A and B have no forward velocity component after the collision.

As to the lateral (up-down) velocity components of A and B, note that these were equal in magnitude and opposite in direction in the laboratory frame. Moreover, this symmetry feature of the velocity diagram cannot be altered by viewing the collision from a rocket frame moving to the right. Therefore the velocities of A and B after the collision, as viewed in the rocket frame, are equal and opposite. This conclusion is payoff No. 1 from arguments based on *symmetry*. Now for payoff No. 2—again achieved by viewing the collision in the rocket frame of reference: In this frame, and *before* the collision, A and B have velocities that are equal in magnitude and opposite in direction. Why? What inconsistency would result if these speeds were *not* equal? *Symmetry* would be violated, as one can see in the following way.

The diagram of the velocity in the rocket frame *after* the collision has *left-right symmetry*. In other words, by looking at the particles separating after the collision it is impossible to tell from what directions the particles arrived at the point of collision.

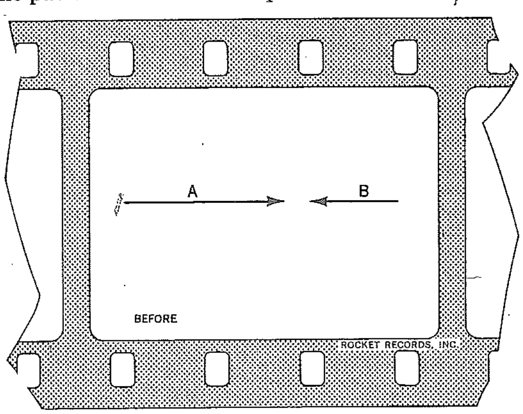


Fig. 56. Rocket record as it would be if, before the collision, particles A and B have different speeds: an incorrect assumption.

Instead of A coming from the left and B coming from the right, A could as well be coming from the right and B from the left (for example, if the viewer went around in back and looked at the collision from the other side).

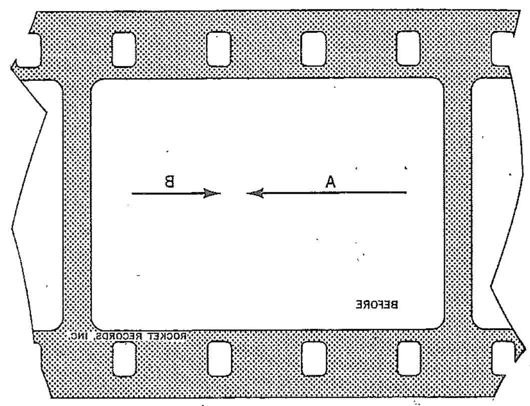


Fig. 57. Rocket record of Fig. 56 looked at from the other side.

But the colliding particles are identical—what is called B in the diagram above could as well have been called A, and conversely:

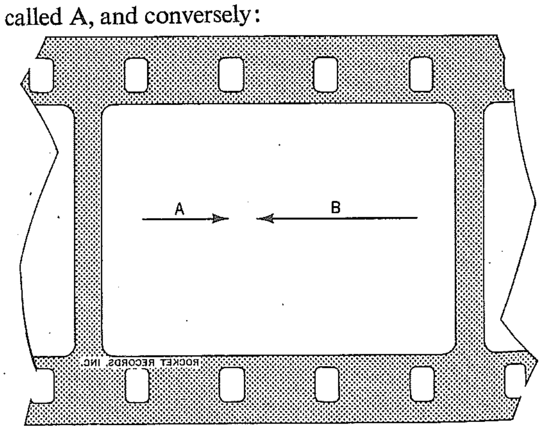


Fig. 58. Rocket record of Fig. 57 with labels A and B for identical balls interchanged.

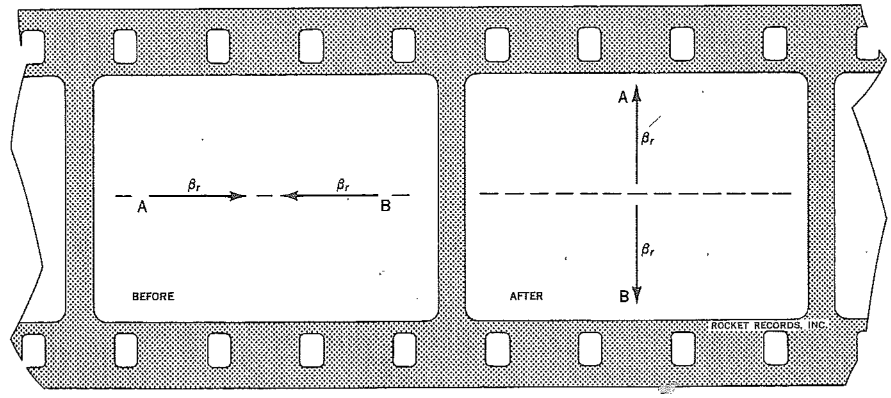


Fig. 59. Conclusion of symmetry arguments: In the rocket frame in which balls A and B have no forward velocity component after the collision, all speeds before the collision and all speeds after the collision have the same value.

Now note that we have in Figs. 56 and 58 two different initial conditions that result in one and the same outcome (Fig. 53). Moreover, these initial conditions differ only in that a suitable increase in the speed of the observing rocket transforms Fig. 56 into the appearance of Fig. 58. But the *outcome* of Fig. 56 cannot continue to look the same as the outcome of Fig. 58 after this increase in the speed of the observer. There is therefore an *inconsistency* in assuming that Fig. 56 and Fig. 58 were different in the first place. To avoid this inconsistency one must conclude that in the rocket frame *A and B have the same speed before the collision*, as drawn in Fig. 55.

Not only do A and B have equal speeds in the rocket frame before the collision—and equal speeds after the collision—but also these speeds before and after the collision are the *same*. If they were not, the following difficulty would arise. (Third use of a symmetry argument—here not symmetry in space but symmetry in time!) Make a moving picture of the collision, develop and print it, and run it *backwards* through the projector. If originally the particles *lost* speed in the collision, they will now be seen to *gain* speed. Such a difference between the two directions of time is a characteristic feature of so-called *irreversible processes*, such as (1) the flow of heat from a hotter object to a cooler one, (2) the aging of an animal, (3) the breaking of an egg, and (4) an inelastic encounter. However, we have limited attention here to *elastic* collisions. Therefore we now accept for study only those events that are *reversible* according to the following definition:

A *reversible* process is one in which it is impossible to distinguish one direction of time from the other by a difference between a film of the process run through the projector in one direction and the same film run through the projector in the other direction.

Because the collision of the two protons is *elastic*, all four speeds in Fig. 59 are *identical*.

This result is very compact and simple. The reasoning leading up to this result can be summarized in a form equally compact and simple. Merely cite these two words: "By symmetry!" Symmetry reasoning of this kind simplifies the analysis of a great variety of physical problems.

The reasoning so far, being based as it is on symmetry considerations, is the same in Newtonian and in relativistic mechanics. The difference between the accounts appears when the now completed rocket-velocity diagram is transformed back to the laboratory frame. In Newtonian mechanics velocities *add as vectors*. Therefore we have only to add the horizontal velocity β_r of the rocket frame after the collision to find the velocities of A and B in the laboratory frame after the collision:

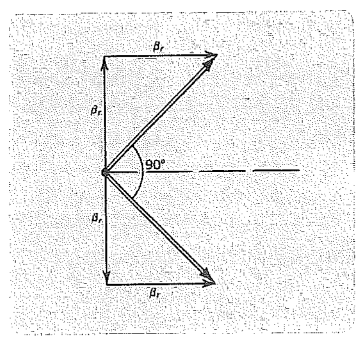


Fig. 60. Newtonian (nonrelativistic) analysis of resultant velocities in the laboratory frame after the collision.

Evidently the angle of separation α is indeed always 90 degrees in Newtonian mechanics, independent of the velocity of the original impact. Not so in relativity!

Show that the incident proton can have a velocity as great as $\beta = 2/7$ without making the angle between v_A and v_B in a symmetric collision depart from the Newtonian value of 90 degrees by as much as 0.01 radian—that is, show that Newtonian mechanics gives good accuracy for a particle with $(2/7)c$ colliding with a particle at rest (or particles with velocity $(1/7)c$ colliding with each other). The results of Ex. 20 may be useful.