Phys 330 Spring 2020

HW #6 Solutions

(1) $V_0 = \frac{1}{2m}\left(\frac{\varepsilon\pi\hbar}{L}\right)^2$

$$a^2 = \frac{mL^2}{2\hbar^2}V_0 = \frac{mL^2}{2\hbar^2}\left(\frac{\varepsilon^2\pi^2\hbar^2}{2mL^2}\right) = \left(\frac{\pi}{2}\varepsilon\right)^2$$

From graphical solution, only 1 bound state for $a < \pi/2$.

$\xi = \frac{L}{2}\frac{\sqrt{2mE}}{\hbar}$ $\quad \varepsilon \ll 1 \Rightarrow \xi \ll 1$

$$\xi \tan\xi \approx \xi^2 = \eta \quad ; \quad \xi^2 + \eta^2 = a^2 = \left(\frac{\pi\varepsilon}{2}\right)^2$$

$$\xi^4 + \xi^2 - \left(\frac{\pi\varepsilon}{2}\right)^2 = 0$$

$$\xi^2 = \frac{1}{2}\left(-1 + \sqrt{1+(\varepsilon\pi)^2}\right) \approx \frac{1}{2}\left(-1+1+\frac{1}{2}(\varepsilon\pi)^2\right)$$

$$\xi = \frac{\varepsilon\pi}{2}$$

$$E = 2\xi^2\left(\frac{\hbar^2}{mL^2}\right) = \frac{\varepsilon^2\pi^2}{2}\left(\frac{\hbar^2}{mL^2}\right) = V_0$$

(2) Beyond classical turning point

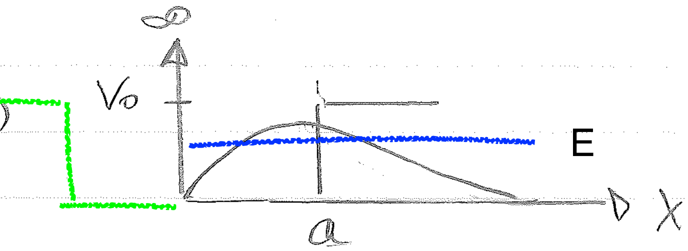
$\psi = Ce^{-qx}$ $\quad q = \frac{\sqrt{2mW}}{\hbar}$ $\quad W \equiv$ work

for $W = 4.7\text{ eV}$ $\quad \lambda = \frac{\hbar c}{\sqrt{2mc^2W}} = \frac{200\text{ eV}\cdot\text{nm}}{\sqrt{2(\frac{1}{2}\times10^6\text{ eV})(4.7\text{ eV})}}$

$= 0.09\text{ nm}$

③

(a)



(b) $\phi(x) = \sin kx \qquad x < a \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

just turns over, $\phi'(a) = 0$

$\cos(ka) = 0 \quad \Rightarrow ka = \pi/2$

$$k^2a^2 = \frac{2mE}{\hbar^2}a^2 = \frac{\pi^2}{4}$$

$$E = \frac{\hbar^2\pi^2}{8ma^2}$$

Compare to particle in finite box, $a = L/2$

$$E = \frac{\hbar^2\pi^2}{2mL^2}$$

which is the minimum energy of the first excited state, as expected from symmetry

see next page

(c) $E_d = \frac{(200\ \text{MeV}\cdot\text{fm})^2}{2\left(\frac{1}{2}10^3\,\text{MeV}\right)}\left(\frac{\pi}{2(1.5\,\text{fm})}\right)^2 = 40\ \text{MeV}$

1D box graphical solution

For 1D box of length L, depth V_0

$k^2 = 2mE/\hbar^2$ and $q^2 = 2m(V_0 - E)/\hbar^2$

Define dimensionless variables $\xi = kL/2$, $\eta = qL/2$ and well strength parameter

$$a^2 = \frac{mV_0L^2}{2\hbar^2}$$

even solutions:

$$\xi \tan \xi = \eta$$

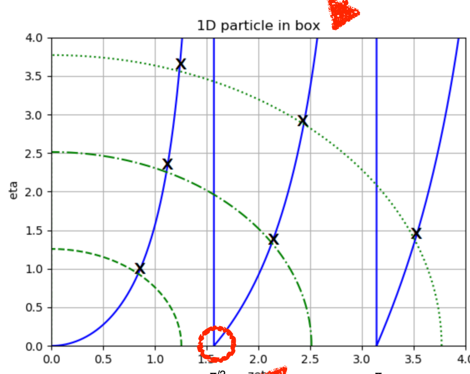
odd solutions:

$$-\xi / \tan \xi = \eta$$

only odd solutions for this problem

intersects with

$$\xi^2 + \eta^2 = a^2$$



minimum enrgy

Figure 1: Graphical solution of 1D box. Three different values of a are drawn. The 'x's mark solutions. Ignore vertical lines which are plotting artifacts.

Intersection values ξ_n have energies

$$E_n = 2\xi_n^2 \frac{\hbar^2}{mL^2} = \left(\frac{2\xi_n}{\pi}\right)^2 \frac{\hbar^2\pi^2}{2mL^2} = \left(\frac{2\xi_n}{\pi}\right)^2 E_1^{\text{box}}$$

In 1D there is always at least one solution no matter how small the strength parameter a.

(4) a) $\phi(x) = \frac{\sqrt{3}}{2}\phi_0 + \left(\frac{1+i}{2\sqrt{2}}\right)\phi_1$

$$\int \phi^*\phi\, dx = \frac{\sqrt{3}}{4}\int \phi_0^*\phi_0\, dx + \left|\frac{1+i}{2\sqrt{2}}\right|^2 \int \phi_1^*\phi_1\, dx = 1$$

since $\int \phi_0^*\phi_1\, dx = 0 = \int \phi_1^*\phi_0\, dx$

and $\left|\frac{1+i}{2\sqrt{2}}\right|^2 = \frac{(1+i)(1-i)}{8} = \frac{1}{4}$

b)

$$\psi(x,t) = e^{-i\hbar\omega t/2}\,\frac{\sqrt{3}}{2}\phi_0 + e^{-i\frac{3\hbar\omega t}{2}}\left(\frac{1+i}{2\sqrt{2}}\right)\phi_1$$

(c) $P_0 = \frac{3}{4}$; $P_1 = \frac{1}{4}$ independent of time

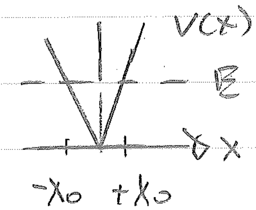
(d) $\langle E\rangle = \frac{3}{4}\left(\frac{1}{2}\hbar\omega\right) + \frac{1}{4}\left(\frac{3}{2}\hbar\omega\right) = \frac{3}{4}\hbar\omega$

$$\langle E^2\rangle = \frac{3}{4}\left(\frac{1}{2}\hbar\omega\right)^2 + \frac{1}{4}\left(\frac{3}{2}\hbar\omega\right)^2 = \left[\frac{3}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{9}{4}\right)\right](\hbar\omega)^2$$
$$= \frac{3}{4}(\hbar\omega)^2$$

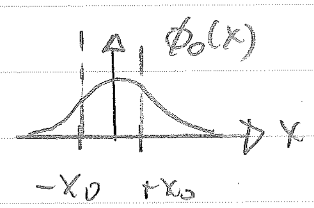
$$(\Delta E)^2 = \langle E^2\rangle - \langle E\rangle^2 = \left(\frac{3}{4} - \frac{9}{16}\right)(\hbar\omega)^2 = \frac{3}{16}(\hbar\omega)^2$$

$\Delta E = \frac{\sqrt{3}}{4}\hbar\omega$ $\langle E\rangle$, ΔE independent of time.

(5)



- ϕ_0 symmetric
- No nodes
- Exp. decay for $|x| > x_0$



Set $p = \hbar/x$ $E(x) = \frac{1}{2m}\left(\frac{\hbar}{x}\right)^2 + bx$

$$\left.\frac{dE}{dx}\right|_{x_m} = 0 = -\frac{\hbar^2}{m x_m^3} + b \Rightarrow x_m = \left(\frac{\hbar^2}{mb}\right)^{1/3}$$

$$E(X_m) = \frac{1}{2} m \hbar^2 \left(\frac{mb}{\hbar^2}\right)^{2/3} + b\left(\frac{\hbar^2}{mb}\right)^{1/3}$$

$$= \frac{1}{2} m^{-1/3} \hbar^{2/3} b^{2/3} + b^{2/3} \hbar^{2/3} m^{-1/3}$$

$$= \frac{3}{2}\left(\frac{\hbar^2 b^2}{m}\right)^{1/3}$$

$\dim[b]$ = energy/length

$\dim\left[\frac{\hbar^2 c^2}{mc^2}\right]$ = energy · length^2

$\dim[E(x_m)] = \left[(\text{energy} \cdot \text{length}^2)\left(\frac{\text{energy}^2}{\text{length}^2}\right)\right]^{1/3}$

= energy

#6

$$\int_{-\infty}^{\infty} \delta(x)\, \phi(x)\, dx = \int \frac{d\Theta}{dx} \phi(x)\, dx$$

$$= \underbrace{\phi(x)\,\Theta(x)\Big|_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{\infty} \Theta(x) \frac{d\phi}{dx}\, dx$$

$$= -\int_0^{\infty} \frac{d\phi}{dx}\, dx = -\phi\Big|_0^{\infty} = \phi(0)$$