Phys 330 Spring 2020

Hw #7 Solutions

① $\psi = c\, e^{-r/a_0}$ is independent of θ, ϕ so $\ell = 0$.

$$V = -\alpha \frac{\hbar c}{r} = -\frac{\hbar^2}{m}\left(\frac{1}{a_0}\right)\left(\frac{1}{r}\right)$$

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r\psi) - \frac{\hbar^2}{m a_0}\left(\frac{1}{r}\right)\psi = E\psi$$

$$\frac{1}{r}\frac{d^2}{dr^2}(r\psi) = \frac{1}{r}\frac{\partial}{\partial r}\left(\psi - \frac{1}{a_0} r\psi\right)$$

$$= \frac{1}{r}\left(-\frac{2}{a_0}\psi + r\left(\frac{1}{a_0}\right)^2\psi\right)$$

giving

cancel

$$-\frac{\hbar^2}{2m}\left(\frac{1}{a_0^2} - \frac{2}{r a_0}\right)\psi - \frac{\hbar^2}{m a_0}\psi = E\psi$$

$$E = -\frac{1}{2}\frac{\hbar^2}{m a_0^2} = -\frac{1}{2} m c^2 \alpha^2$$

normalization on next page.

② $$P(r_0) = \int_0^{r_0} r^2 dr \int d\Omega \left(\frac{1}{\pi a_0^3}\right) e^{-2r/a_0}$$

For $r_0 \ll a_0$, $e^{-r2/a_0} \approx 1$ and

$$P(r_0) \cong 4\pi\left(\frac{1}{\pi a_0^3}\right)\frac{1}{3} r_0^3 = \frac{4}{3}\left(\frac{r_0}{a_0}\right)^3$$

$$\cong \left(\frac{.10^{-6}\ \text{nm}}{.05\ \text{nm}}\right)^3 = \left(\frac{1}{5}\right)^3 10^{-4\cdot 3} \cong 10^{-14}$$

$$P(r > a_0) = \int_{a_0}^{\infty} 4\pi r^2 dr \left(\frac{1}{\pi a_0^3}\right) e^{-2r/a_0}$$

$x = 2r/a_0\big|_{a_0} = 2$

$$= \frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^3 \int_2^{\infty} x^2 e^{-x} dx = \frac{1}{2}(1.35) \approx 0.7$$

HW # 7, Prob 1

normalize $\psi_{10} = Ce^{-r/a_0}$ hydrogen ground state wave function

$$1 = \int |\psi_{10}|^2 d^3r \Rightarrow = C^2 \int_0^\infty r^2 dr\, e^{-2r/a_0} \underbrace{\int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi}_{4\pi}$$

$$C^{-2} = 4\pi \left(\frac{a_0}{2}\right)^3 \underbrace{\int_0^\infty x^2 e^{-x} dx}_{2!} = \pi a_0^3$$

$$C = \left(\frac{1}{\pi a_0^3}\right)^{1/2}$$

(3) $\Delta E(2\to 1) = -\mu c^2 \alpha^2 \left(\frac{1}{4} - 1\right)$

$$\Delta E_D / \Delta E_H = \frac{\mu_D}{\mu_H} = \left(\frac{\frac{m_e(2m_N)}{m_e + 2m_N}}{\frac{m_e m_N}{m_e + m_N}} \right)$$

$$= \frac{2(m_e + m_N)}{m_e + 2m_N} = 2\left(\frac{m_e/m_N + 1}{m_e/m_N + 2}\right) = 1 + 2.5 \times 10^{-4}$$

$$\frac{m_e}{m_N} = \frac{0.5\ \text{meV}}{1000\ \text{meV}} = \frac{1}{2} \times 10^{-3}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta E}{E} = 2.5 \times 10^{-4}$$

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~~5~~ Electron as sphere of radius r_0 spinning with angular frequency Ω

$$S = I\omega = \frac{3}{5} M r_0^2 \omega$$

$$\omega = \frac{S}{I} = \frac{\hbar/2}{\frac{2}{5} m r_0^2} = \frac{5}{4} \frac{\hbar c}{mc^2}\left(\frac{c}{r_0^2}\right)$$

take $r_0 = 10^{-18}\, m = 10^{-9}\, nm$

$$\omega = \frac{5}{4} \cdot \frac{(200\ eV \cdot nm)(3 \times 10^{8+9}\ nm/s)}{\frac{1}{2} \times 10^{6}\ eV\ (10^{-9}\ nm)^2}$$

$$= \frac{5}{4} \cdot 4 \cdot 3 \cdot 10^{2+17-6+18}\ s^{-1} = 1.5 \times 10^{32}\ s^{-1}$$

Point on surface moves with speed

$$v = \omega r_0 = (1.5 \times 10^{32}\ s^{-1})(10^{-18} m)$$
$$= 1.5 \times 10^{14}\ m/s \gg c$$

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~~6~~ Atoms inside Stern-Gerlach non-uniform gradient will be accelerated as

$$m\ddot{z} = -\mu \frac{dB}{dz} = -\left(\frac{ge}{2m}\right) S_z \frac{dB}{dz} = \pm\left(\frac{e\hbar}{2m}\right)\frac{dB}{dz}$$

$$= \pm \mu_B \left(\frac{dB}{dz}\right).$$

after a distance L at speed $v = \sqrt{\frac{2E_K}{m}}$ deflection is

$$z = \frac{1}{2} a_z \left(\frac{L}{v}\right)^2 = \frac{1}{2} a_z \left(\frac{L}{\sqrt{2E_K/m}}\right)^2 = \frac{1}{4} m a_z \frac{L^2}{E_K}$$

Splitting between two beams

$$\Delta = 2z = \frac{1}{2}\mu_B \frac{L^2}{E_k}\frac{dB}{dz}$$

$$\langle E_k \rangle = 2kT \quad \text{at } T = 10^{3}\,K$$

$$\frac{dB}{dz} = \left.\frac{4(kT)z}{\mu_B L^2}\right|_{\substack{z = 10^{-3}\,m \\ L = \frac{1}{2}10^{-1}\,m}} = \frac{4(8.62\times10^{-7}\,eV)\,10^{-3}\,m}{(5.79\times10^{-5}\,eV/T)\left(\frac{1}{2}10^{-1}\right)^2}$$

$$= \frac{4}{5.79}\frac{(8.62)}{1}\,10^{2}\,T/m = 2.38\times10^{-3}\,T/m$$

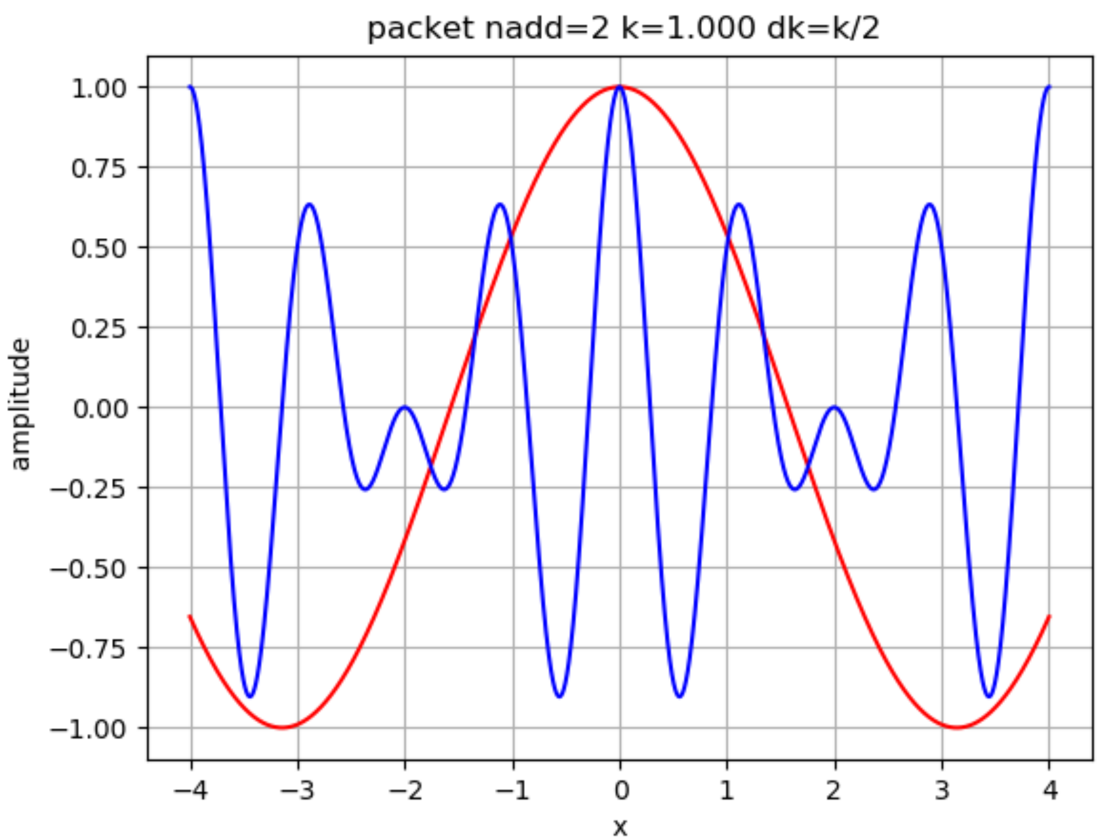
Date: March 8, 2020 at 11:50 AM
Topic: wave-packets

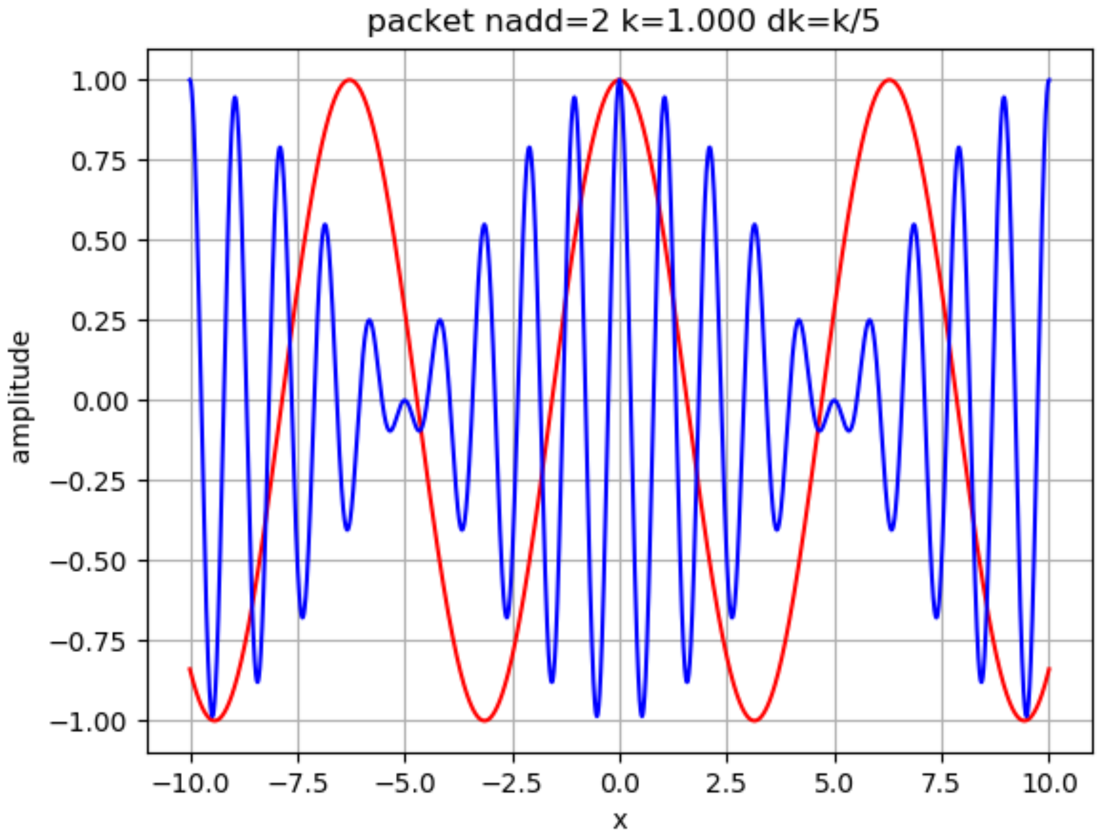
6 ~~#7)~~

The relation is easily shown with trig. identities or by taking the real part after writing as complex exponentials.

Take Delta_x where the cosine(Delta_k x) goes to zero,

Delta_x = pi/Delta_k then Delta_x Delta_k = pi

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0.50
0.25
0.00
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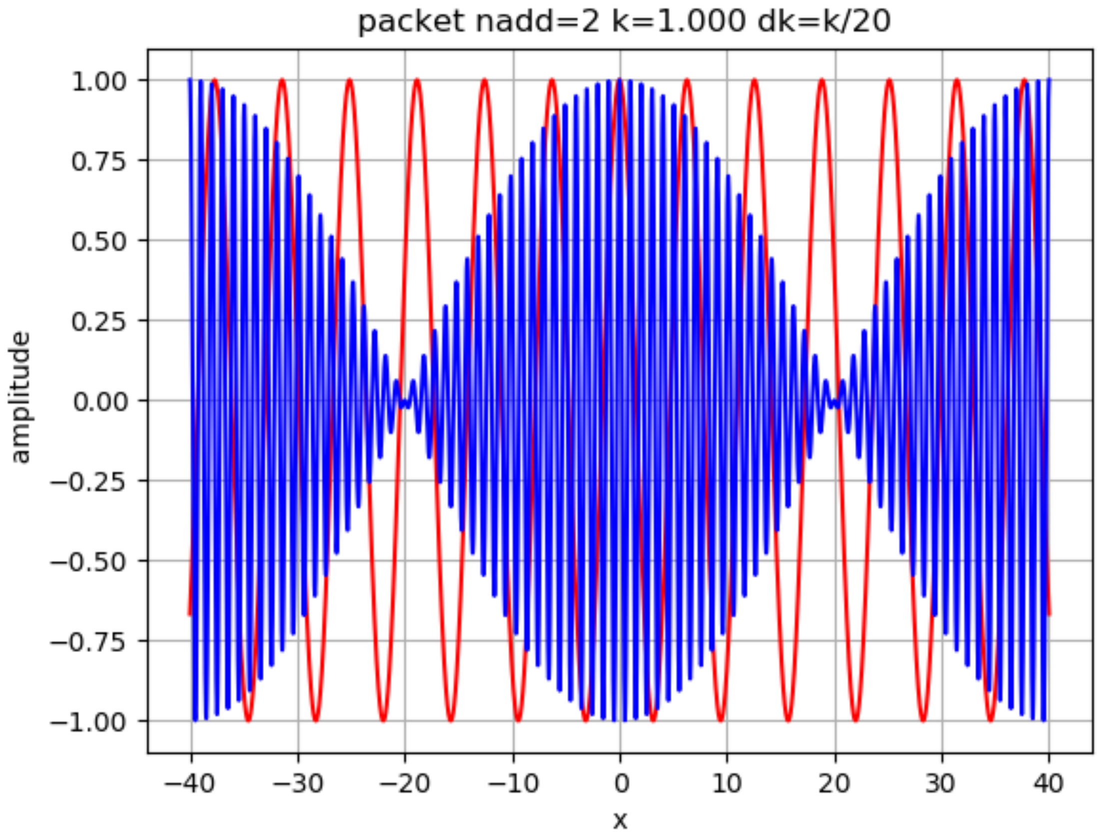
5.0
7.5
10.0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0.50
0.25
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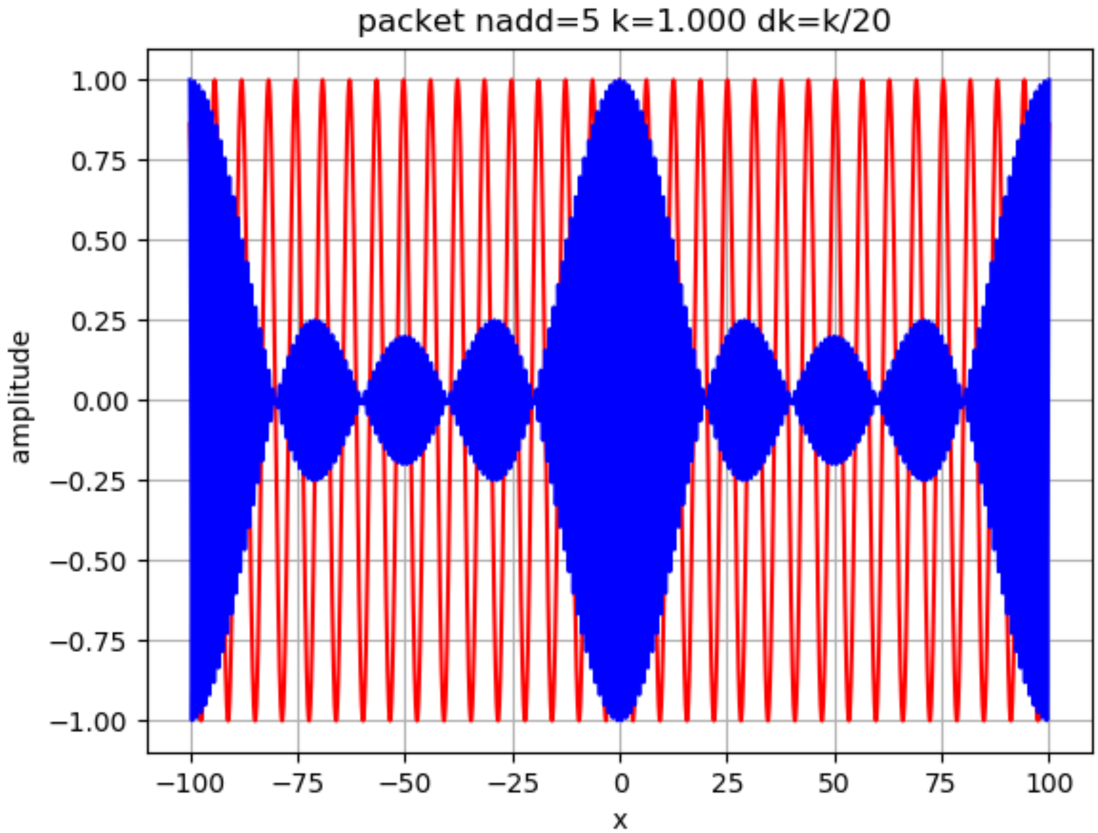
−30
−20
−10
0
10
20
30
40
x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−1.00
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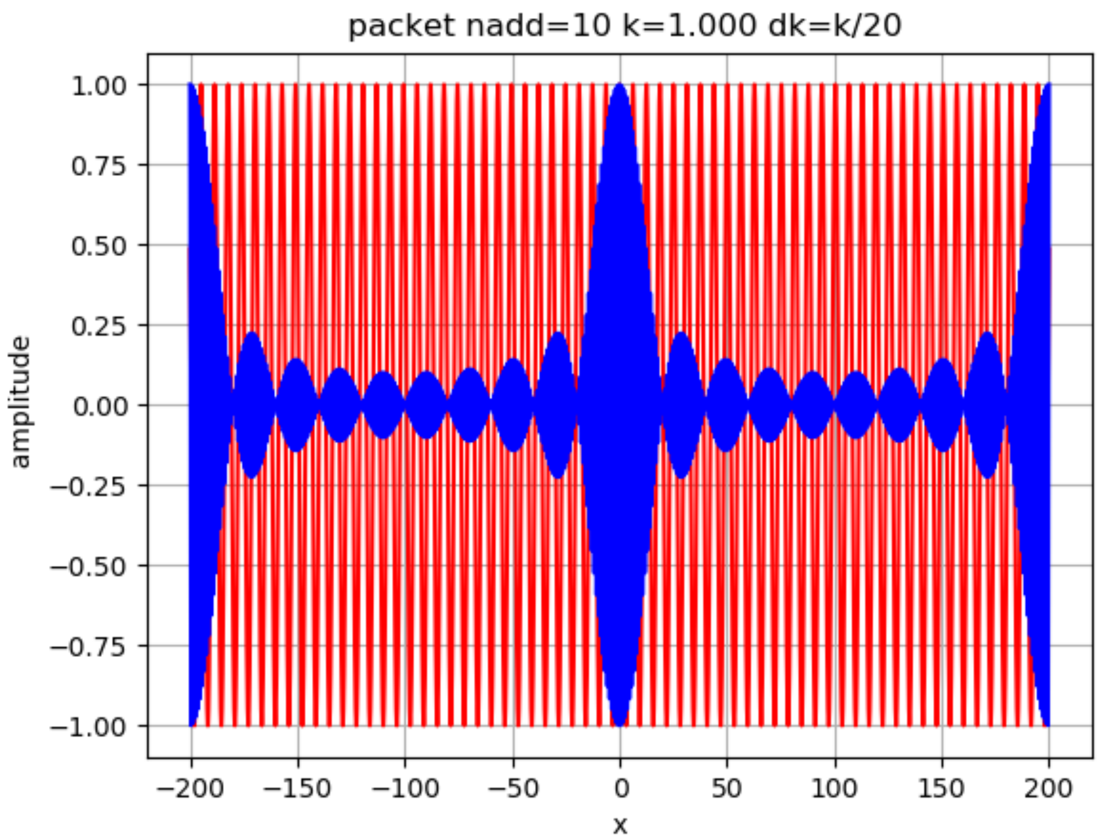
−75
−50
−25
0
25
50
75
100
x



As the sum goes over to an integral, we get an isolated wave packet. A smoother wave packet would result from integrating over a smoother function of k in the range $k - \Delta k < k < k + \Delta k$