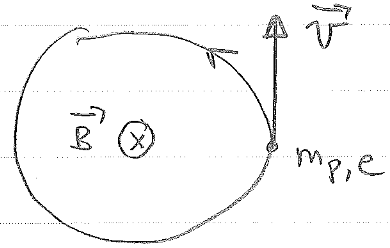
Phys 330 Spring '20

Lecture #1 : Lorentz Transformation

① E.O. Lawrence & the cyclotron



$$\vec{F} = e\,\vec{v} \times \vec{B} = m\,\dot{\vec{v}}$$

for uniform circular motion, $a = v^2/r$ and so

$$m\left(\frac{v^2}{r}\right) = e v B$$

$$mv = \boxed{p = eBr}$$

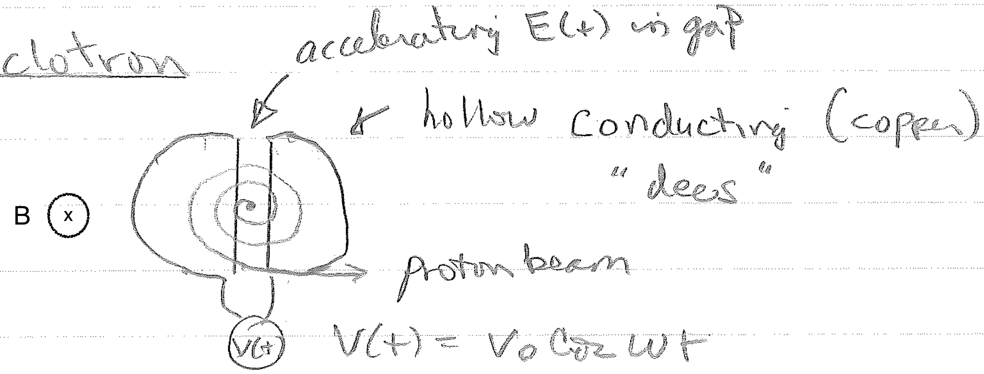
* holds in SR

frequency $f = \frac{v}{2\pi r} = \left(\frac{eBr}{m}\right)\left(\frac{1}{2\pi r}\right) = \frac{eB}{2\pi m}$

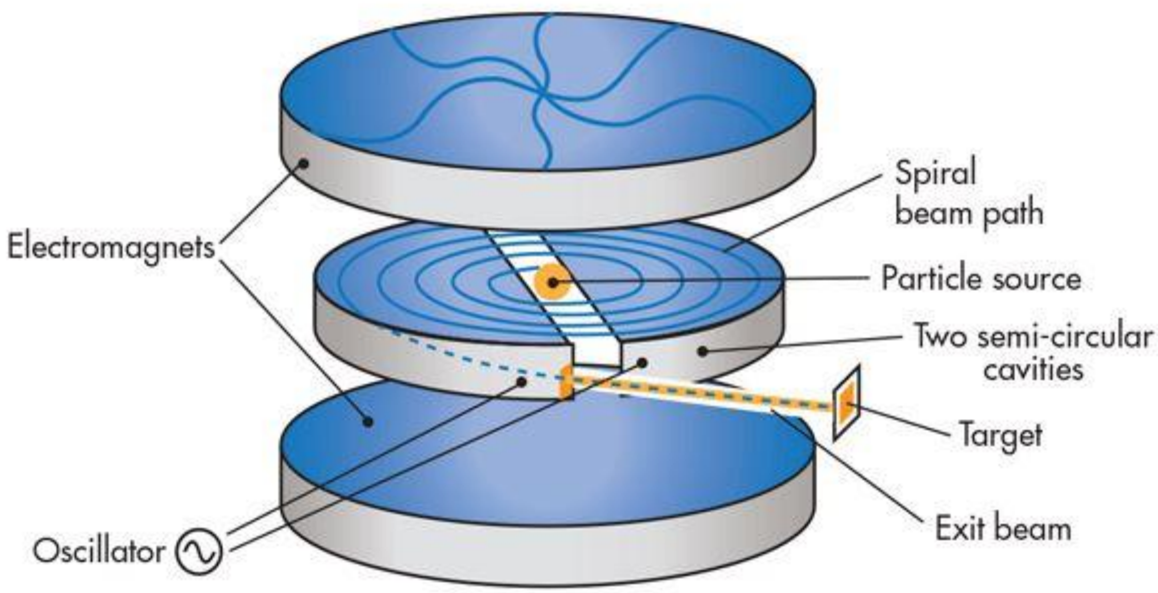
$$\omega = 2\pi f = \frac{eB}{m} = \left(9.6 \times 10^7 \text{ rad/s/T}\right) B$$

⇒ f independent of radius r

the cyclotron



Cyclotrons



(Image courtesy of Symmetry Magazine)

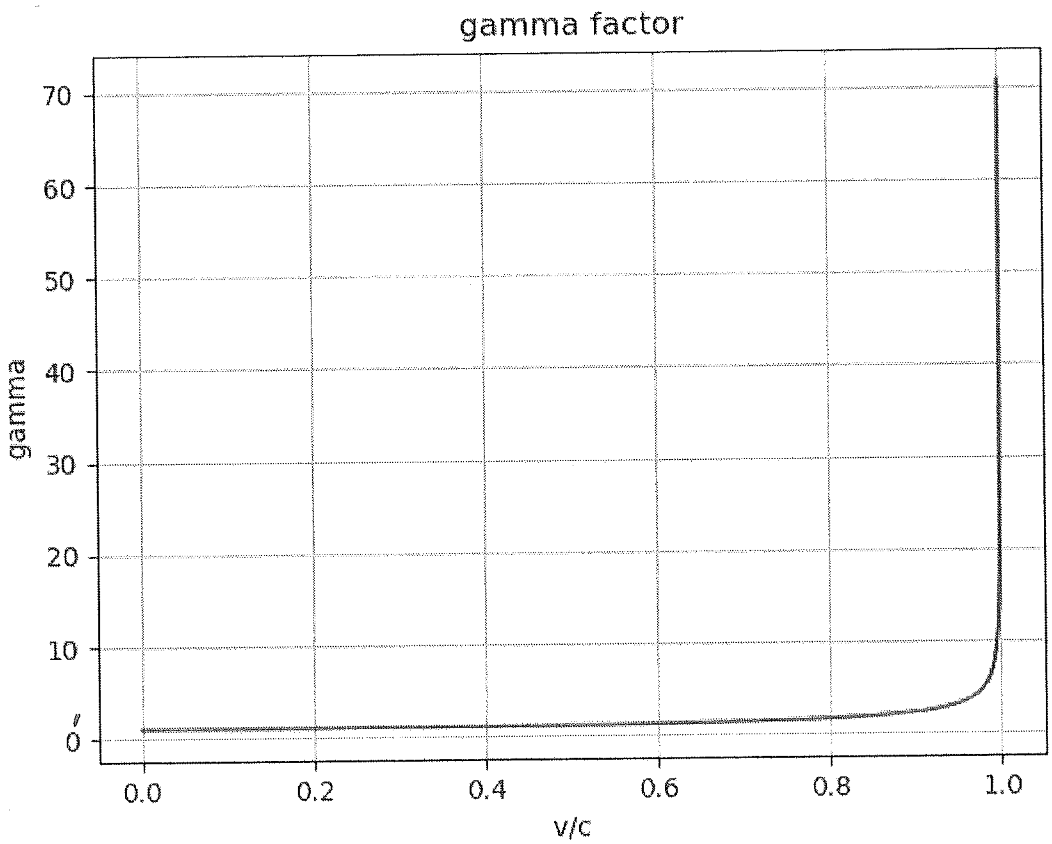
Experimentally, E_k^{max} = few x10 MeV
corresponding to proton speed

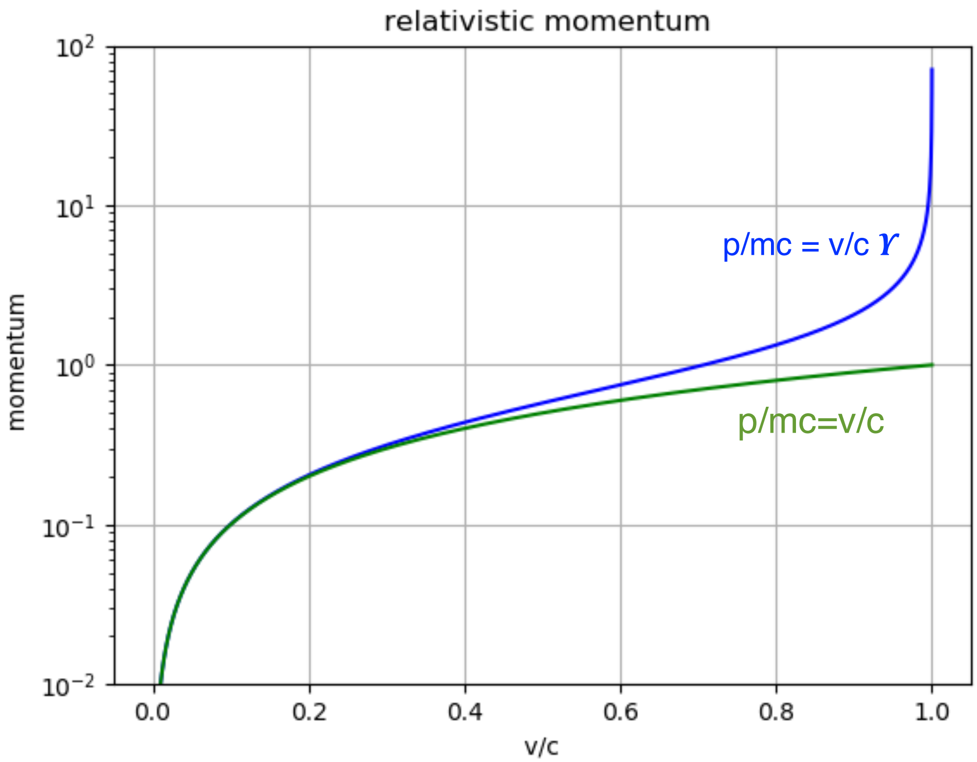
$$V_{max} = c\sqrt{\frac{2E_k^{max}}{M_pc^2}} \cong \sqrt{\frac{2(20\,\text{MeV})}{10^3\,\text{MeV}}} \approx 0.2c$$

(2) SR - speed of light $c = \left(\frac{3}{10}\right)$ m/ns
constant in all inertial frames.

$$\vec{p} = m\gamma\vec{v}$$

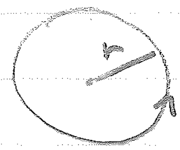
$$\gamma = \left[1-\left(\frac{v}{c}\right)^2\right]^{-1/2} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{1}{4!}\left(\frac{v}{c}\right)^4 + \cdots$$





relativistic momentum
10^2
10^1
10^0
10^{-1}
10^{-2}
momentum
0.0
0.2
0.4
0.6
0.8
1.0
v/c
p/mc = v/c γ
p/mc=v/c

Cyclotron breaks down because rotation frequency no longer constant.



$$2\pi f \cdot r = c$$
$$f = c/2\pi r$$

Synchrotron - rather than increase r, increase B.

<u>Consequences of S.R.</u>:

① "moving clocks run slow" $\Delta t = \gamma \Delta \tau$
  τ is <u>proper time</u>, measured in rest frame of clock

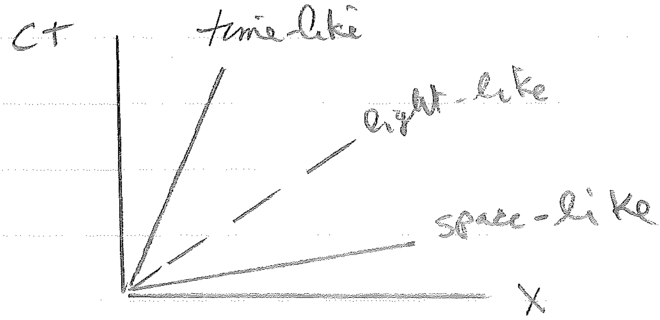
② "moving meter sticks contracted" $\Delta l = \frac{\Delta S}{\gamma}$
  S is <u>proper length</u> measured in rest frame of meterstick

③ simultaneity is frame dependent

④ No perfectly rigid objects
  shocks cannot travel faster than c.

<u>Invariant interval</u>: $\vec{\Delta r} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$ $|\vec{\Delta r}|^2 = \vec{\Delta r} \cdot \vec{\Delta r}$

$$c^2(\Delta t)^2 - |\vec{\Delta r}|^2 = \begin{cases} c^2 \Delta t^2 & \text{timelike} \\ 0 & \text{light-like} \\ -(\Delta S)^2 & \text{space-like} \end{cases}$$



Matrix multiplication

Euclidean vector $\vec{A} = \sum_{i=1}^{3} A^i \hat{e}_i \xrightarrow{\text{represented by}} \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}$

component basis

Dot product

$$\vec{B}\cdot\vec{A} = \left(\sum_{j=1}^{3} B^j \hat{e}_j\right)\cdot\left(\sum_{i=1}^{3} A^i \hat{e}_i\right)$$

Basis is orthonormal – $\hat{e}_j \cdot \hat{e}_i = \delta_{ij}$ Kronecker delta

$$\equiv \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

$$\vec{B}\cdot\vec{A} = \sum_{\substack{i=1\\j=1}}^{3} B^j A^i \delta_{ij} = \sum_{i=1}^{3} B^i A^i$$

$$= \left(B^1A^1 + B^2A^2 + B^3A^3\right) \quad \text{a scalar}$$

$$= \underset{\text{row}}{\left(B^1, B^2, B^3\right)} \underset{\text{column}}{\begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}}$$

Matricies multiply in similar row x column fashion. Consider 2x2 for simplicity

$$\bar{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = [M_{ij}]$$

row   column

$$\bar{O} = \bar{M}\cdot\bar{N} \qquad O_{ij} = \sum_{k=1}^{2} M_{ik} N_{kj}$$

summed index

row dot column

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} =$$

$$\rightarrow \begin{pmatrix} M_{11}N_{11} + M_{12}N_{21}, & M_{11}N_{12} + M_{12}N_{22} \\ M_{21}N_{11} + M_{22}N_{21}, & M_{21}N_{12} + M_{22}N_{22} \end{pmatrix}$$

$$= \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}$$

<u>matrix multiplication does not commute.</u>

$$\bar{M}\cdot\bar{N} \neq \bar{N}\cdot\bar{M}$$

Matricies arise naturally in systems of linear equations (linear algebra)

$$B^1 = m^1_1 A^1 + m^1_2 A^2$$
$$B^2 = m^2_1 A^1 + m^2_2 A^2$$

$$\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} = \begin{pmatrix} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$$

$$\vec{B} = \bar{M} \circ \vec{A}$$

inverse matrix $\bar{M}^{-1} \cdot \bar{m} = \bar{I}$ identity $\bar{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\bar{m}^{-1} = \frac{1}{\underbrace{(m^1_1 m^2_2 - m^2_1 m^1_2)}_{\det \bar{m} \text{ determinant}}} \begin{pmatrix} m^2_2 & -m^1_2 \\ -m^2_1 & m^1_1 \end{pmatrix}$$

then $\vec{A} = \bar{M}^{-1} \cdot \vec{B}$

Easy to do matrix inversions numerically: Python example

In [9]:
```
import numpy as np
```

In [11]:
```
A = np.array([
    [6, 1, 1, 1],
    [4, -2, 5, 1],
    [2, 8, 7, 1 ],
     [1,2,3,4]])
```

In [12]:
```
print(A)
```

```
[[ 6  1  1  1]
 [ 4 -2  5  1]
 [ 2  8  7  1]
 [ 1  2  3  4]]
```

In [13]:
```
B= np.linalg.inv(A)
```

In [14]:
```
print(B)
```

```
[[ 0.17399267  0.00457875 -0.01007326 -0.04212454]
 [ 0.05860806 -0.13003663  0.08608059 -0.003663  ]
 [-0.11904762  0.1547619   0.05952381 -0.02380952]
 [ 0.01648352 -0.0521978  -0.08516484  0.28021978]]
```

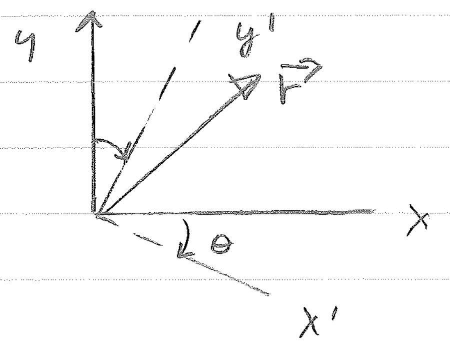
In [15]:
```
print(np.matmul(A,B))
```

```
[[ 1.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00]
 [-1.11022302e-16  1.00000000e+00  5.55111512e-17 -5.55111512e-17]
 [-5.55111512e-17  0.00000000e+00  1.00000000e+00 -5.55111512e-17]
 [ 0.00000000e+00 -5.55111512e-17  5.55111512e-17  1.00000000e+00]]
```

In [ ]:

In [ ]:

What is a Euclidean vector?



passive left-handed rotation

$$\vec{r} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = x\,\hat{e}_x + y\,\hat{e}_y$$

↑ represented by

$$\vec{r} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = x'\,\hat{e}_x' + y'\,\hat{e}_y'$$

$\boxed{\text{often we use "=" instead of "→". Here we emphasis that the vector } \vec{r} \text{ is the same geometric object}}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} C & -S \\ +S & C \end{pmatrix}}_{\bar{R}(\theta\,\hat{e}_z)} \begin{pmatrix} x \\ y \end{pmatrix}$$

$C \equiv \cos\theta$

$S \equiv \sin\theta$

Right-hand rotation of coordinates

Coordinates (vector components) rotate in <u>opposite</u> direction of basis vectors (coordinate system)

Leaves norm $|\vec{r}|$ invariant.

$$\vec{r}\cdot\vec{r} = x^2 + y^2 = x'^2 + y'^2$$

Successive rotation $\bar{R}(\theta_1\hat{e}_z)\,\bar{R}(\theta_2\hat{e}_z) = \bar{R}((\theta_1+\theta_2)\hat{e}_z)$

follow from trig. identities

$$\begin{pmatrix} \cos(\theta_1+\theta_2), & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2), & \cos(\theta_1+\theta_2) \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

Note: rotations along non-parallel axes <u>do not commute</u>

$$\bar{R}(\theta_1 \hat{e}_x)\,\bar{R}(\theta_2 \hat{e}_z) \neq \bar{R}(\theta_2 \hat{e}_z)\,\bar{R}(\theta_1 \hat{e}_x)$$

<u>Minkowski 4-vector</u>

event: place + time

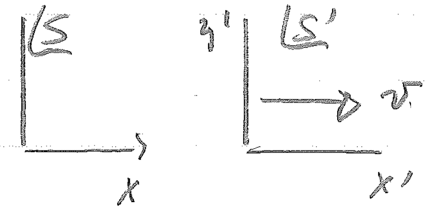
$$\tilde{x} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$ with dot-product

$$\tilde{x}\cdot\tilde{x} = (ct)^2 - \vec{r}\cdot\vec{r}$$

↑ Minkowski metric

invariance of dot-product gives Lorentz transformation (LT) of coordinates in frame S to frame S'

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$$ and $y' = y$, $z' = z$

where 

$$\beta = v/c$$
$$\gamma = [1-\beta^2]^{-\frac{1}{2}}$$

frame S' moves with speed v with respect to S along common direction $+\hat{x} = +\hat{x}'$

Call this LT a "boost" in $+\hat{x}$ direction.

Boost Angle

Analogous to rotation angle we can introduce a hyperbolic "boost" angle.

$$\gamma^2 - \beta^2\gamma^2 = 1 = \cosh^2\theta - \sinh^2\theta = 1$$

$\cosh\theta = \gamma$

$\sinh\theta = \beta\gamma$ $\boxed{\tanh\theta = \beta}$

then LT is boost matrix

$$\bar{B}(\theta\hat{e}_x) = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$

Easy to show using hyperbolic identities,

$$\bar{B}((\theta_1+\theta_2)\hat{e}_x) = \bar{B}(\theta_1\hat{e}_x)\,\bar{B}(\theta_2\hat{e}_x)$$

Note - not only do boosts in different directions not commute, they are not even a boost! They are a boost & rotation.

Addition of velocities follow from hyperbolic identity

$$\tanh(\theta_1+\theta_2) = \frac{\tanh(\theta_1) + \tanh(\theta_2)}{1 + (\tanh\theta_1)(\tanh\theta_2)}$$

usefull Taylor expansions

$$\sinh\theta = \theta + \frac{\theta^3}{3!} + \cdots$$

$$\cosh\theta = 1 + \frac{\theta^2}{2!} + \cdots$$

$$\tanh\theta = \theta - \theta^3/3 + \cdots$$