Phys 330 Spring 2020

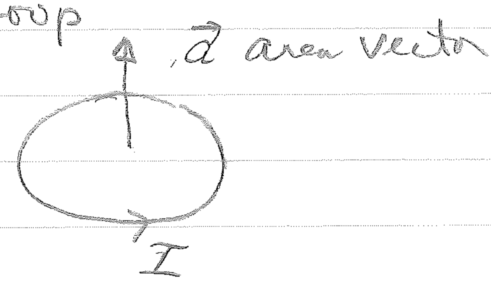
Lecture # 12 - Spin

Electron intrinsic spin

intrinsic means there is no kinetic energy term in Hamiltonian (energy)

classical current loop

$$\vec{\mu} = I\,\vec{a}$$



rewrite in terms of orbital (spin) angular momentum $\vec{S}$!

$$S = mr^2\omega$$

$$I = q\frac{v}{2\pi r} = q\frac{\omega}{2\pi}$$

$$\mu = \frac{q\,\omega}{2\pi}\left(\pi r^2\right) = \left(\frac{q}{2m}\right)S$$

$$\vec{\mu} = \frac{q}{2m}\vec{S}$$

Classical spinning sphere with uniform surface charge

$$\vec{\mu} = \frac{q}{2m}\underbrace{\left(\frac{5}{3}\right)}\vec{S}$$

gyromagnetic ratio $\equiv g$

Torque $\vec{N}$ in magnetic field

$$\vec{N} = \frac{d\vec{S}}{dt} = \vec{\mu} \times \vec{B} = \frac{q}{2m} g\, \vec{S} \times \vec{B}$$

and time derivative $\frac{d\vec{S}}{dt} = \vec{S} \times \vec{\omega}$

from which we see $\omega_L = g \frac{q}{2m} B$ Larmor frequency

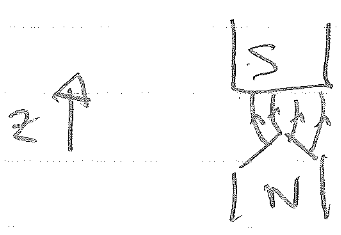
<u>Potential Energy</u> $V = -\vec{\mu} \cdot \vec{B}$

$\vec{\mu}$ points along $\vec{B}$ has lowest energy

In non-uniform $\vec{B} = B(z)\hat{z}$
dipole experiences force

$$F_z = -\frac{d}{dz} V = -\mu_z \frac{dB}{dz}$$

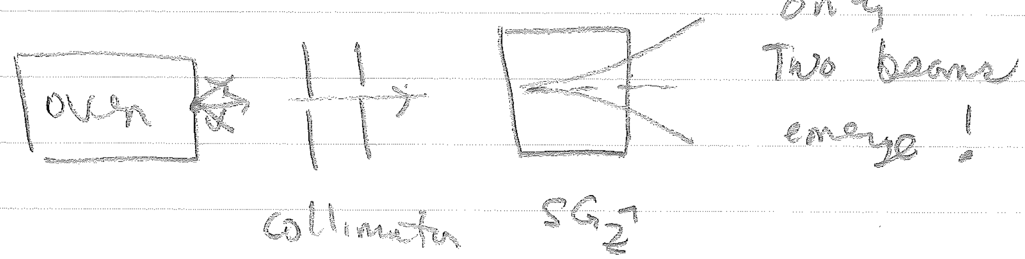
Classically, $\mu_z \equiv \vec{\mu} \cdot \hat{z}$ takes on continuous values. <u>Stern-Gerlach Experiment</u>



$\frac{dB}{dz} < 0$

refer to this measurement as $SG_{\vec{z}}$

Silver atoms from oven have single e^- in outer shell, so have spin of electron.



Interpretation: $\vec{S}$ quantized

$$S_z = \pm \frac{\hbar}{2}$$

This new degree of freedom is described mathematically by two component complex spinor:

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \qquad \chi_\pm \text{ complex number}$$

complex conjugate transpose

$$\chi^\dagger \chi = (\chi_+^*, \chi_-^*) \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = |\chi_+|^2 + |\chi_-|^2 = 1$$

two beams $\Rightarrow$ quantum number $S = \frac{1}{2}$

states $(2S+1) = 2$

allowed values $-m_s \leq S \leq m_s$

$$m_s = \pm \frac{1}{2}$$

This works the same as for orbital angular momentum because (math) group theory describes rotations by $SU(2)$

interlude ...

SU(2) rotates vector space by angles $\theta_x, \theta_y, \theta_z$.

$\hat{T}_i$ represented by:

(a) differential operators acting on vector space of orthogonal functions

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}, \text{ etc.}$$

(b) D×D dimensional matrices acting D dimensional vectors

Spinor D = 2 lowest, defining acts on $\chi = \binom{a}{b}$ a, b complex

Vector D = 3 acts on Euclidean vectors $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ x, y, z real

To rotate system, rotate everything by same angles $\theta_x, \theta_y, \theta_z$ according to representation

So quantum number l, m_l ; s, m_s have same behavior because group is the same SU(2)

$\boxed{l = 0, 1, 2, \ldots \infty \text{ orbital has no limit}}$

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$$
$$\hat{L}_z Y_{lm} = \hbar m Y_{lm}$$
$$\hat{S}^2 \chi = \hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1)\chi$$

"spin ½" $\hat{S}_z \chi = \pm \frac{1}{2}\chi$ highest eigenvalue is ½

S.R. + Q.M $\Rightarrow$ Dirac equation
predicts electron spin with gyromagnetic ratio

$$g_e = 2(1 + a_e)$$

$\uparrow$ $\sim 10^{-3}$ tested to $1/10^{9}$ precision

true for any "point-like", elementary fermion.

$$\vec{\mu}_e = \frac{e}{2m} g \vec{S} = \frac{e}{m} \vec{S}$$

$\pm \frac{\hbar}{2}$ eigenvalue

$$\vec{\mu}_e \cdot \vec{B} = \pm \left(\frac{e\hbar}{2m}\right) B_z$$

$\uparrow$ eigenvalue m_s

$$\frac{e\hbar}{2m} \equiv \text{Bohr magneton} = 5.788 \times 10^{-5}\ \text{eV}/T$$

Orbital angular momentum commutators

$$\hat{L}_x = y\hat{P}_z - z\hat{P}_y$$
$$\hat{L}_y = z\hat{P}_x - x\hat{P}_z$$
$$\hat{L}_z = x\hat{P}_y - y\hat{P}_x$$

$$[\hat{L}_x, \hat{L}_y] = [y\hat{P}_z - z\hat{P}_y,\ z\hat{P}_x - x\hat{P}_z]$$
$$= [y\hat{P}_z, z\hat{P}_x] + \underbrace{[y\hat{P}_z, -x\hat{P}_z]}_{=0}$$
$$+ \underbrace{[-z\hat{P}_y, z\hat{P}_x]}_{=0} + [-z\hat{P}_y, -x\hat{P}_z]$$
$$= y\hat{P}_x \underbrace{[\hat{P}_z, z]}_{-i\hbar} + x\hat{P}_y \underbrace{[z, \hat{P}_z]}_{i\hbar} = i\hbar[x\hat{P}_y - y\hat{P}_x]$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \quad \text{and cyclic}$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$

Or $[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \varepsilon_{ijk}\hat{L}_k$

$\varepsilon_{ijk} = 0$ if any 2 indices the same

$\varepsilon_{123} = 1$ completely

$\varepsilon_{213} = -1$ antisymmetric

From algebra of commutators alone,

$$\hat{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

$$\hat{L}_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad -\ell \le m \le \ell$$

Same algebra (group SU(2)) exists for 2-dim spinors acted on by spin operators

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\hat{S}_x, \hat{S}_y] = \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right]$$

$$= i\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\hat{S}_z$$

$$\chi_+^z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (\chi_-^z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \hat{S}_z \chi_\pm^z = \pm\frac{\hbar}{2}\chi_\pm^z$$

$$\hat{S}^2 = \frac{\hbar^2}{4}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right)$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3\hbar^2}{4} I$$ unit 2x2

$$\hat{S}^2 \chi_\pm^z = \frac{3}{4}\hbar^2 \chi_\pm^z$$

$$\frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right)$$ rule for quantum number $s = \frac{1}{2}$

Modern Physics 330: Lec 12 – More on Spin

Lets explore how a spin-1/2 particle like the electron behaves when the spin is measured with respect to different directions. The spin wave function χ is called a spinor, and is a two-comonent complex vector. Just as for a Euclidean vector, a spinor can be written in any basis. For example, in the z basis,

$$\chi = \begin{pmatrix} z^+ \\ z^- \end{pmatrix}_z$$

Here $z^\pm$ are ampitudes to be measured with spin eigenvalues $\pm\hbar/2$ with respect to the $\hat{z}$ direction. Squaring (amplitude $\times$ amplitude*) the amplitudes gives the probabilities. (Here * means complex-conjugation.) For example, the electron in the state

$$\chi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}_z$$

has equal probability 1/2 to be measured with spin up or spin down. Notice the normalization is $(z^+)^\star(z^+) + (z^-)^\star(z^-) = 1$.

Or in a basis rotated with respect the z-axis into a direction $\hat{n}$ making an angle θ with respect to the $\hat{z}$ axis.

$$\chi = \begin{pmatrix} n^+ \\ n^- \end{pmatrix}_n$$

The components of the spinor are related by the spinor rotation matrix,

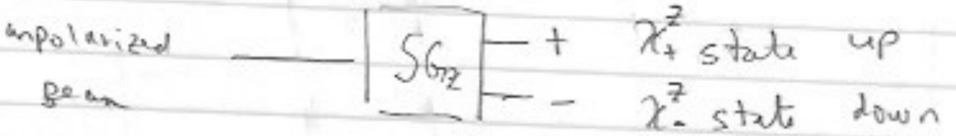
$$\begin{pmatrix} n^+ \\ n^- \end{pmatrix}_n = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} z^+ \\ z^- \end{pmatrix}_z$$

Notice the screwy $\theta/2$, meaning you don't get back the same state until rotating by 4π!

Symbol for Stern Gerlach device with $\vec{B}$ along $\hat{n}$ direction

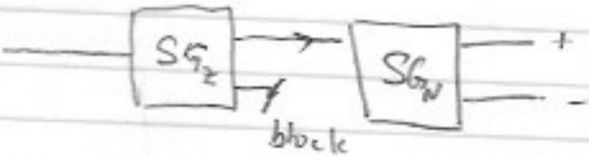
$\boxed{SG_{\hat{n}}}$

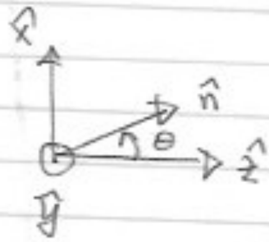
Silver atoms (spin ½) from oven are unpolarized



$$\chi_+^z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^z \qquad \chi_-^z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^z$$

suppose we select spin up state from $\boxed{SG_z}$ and then measure spin along $\hat{n}$ direction





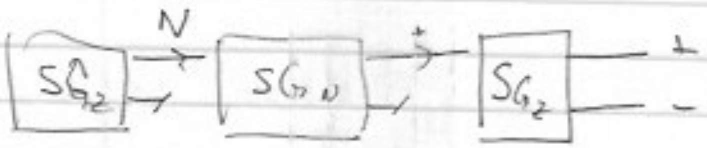
amplitude to be up. down along $\hat{n}$ direction

$$\begin{pmatrix} n_+ \\ n_- \end{pmatrix}_n = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_z$$

$$n_+ = \cos\theta/2 \quad \rightarrow \quad n_+ = \frac{1}{\sqrt{2}} \quad P_+ = |n_+|^2 = \frac{1}{2}$$
$$n_- = -\sin\theta/2 \quad \theta = \pi/2 \quad n_- = -\frac{1}{\sqrt{2}} \quad P_- = |n_-|^2 = \frac{1}{2}$$

probabilities

Suppose we select n_+ beam and then measure along $\hat{z}$ again



use inverse rotation matrix

$$\begin{pmatrix} z_+ \\ z_- \end{pmatrix}_z = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}_z$$

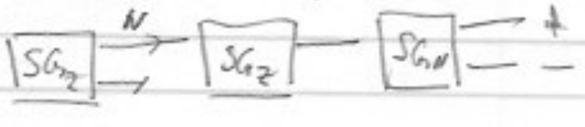
Starting with N atoms from first $\boxed{SG_z}$

Number from each beam is product of probabilities

$$N_+ = N(\cos\theta/2)^2(\cos\theta/2)^2$$

$$N_- = N(\cos\theta/2)^2(\sin\theta/2)^2$$

If we reversed order of last two measurements



Second $\boxed{SG_z}$ does nothing and we get

$$N_+' = N(1)(\cos\theta/2)^2$$
$$N_-' = N(1)(\sin\theta/2)^2$$

Order of measurements matters.
Corresponding QM. operators do not commute.

$$\hat{S}_z\hat{S}_N \neq \hat{S}_N\hat{S}_z$$