Physics 330 Spring 2020

## Lec 16: Statistical Physics II: Quantum

Classical Boltzmann distribution for molecules in gas

$$\overline{n}_{MB}(\varepsilon) = \frac{N}{kT} e^{-\varepsilon/kT} \quad ; \quad \int_0^\infty \overline{n}_{MB}(\varepsilon)\, d\varepsilon = N$$

$\varepsilon$ continuous and molecules distinguishable.

Validity - mean free path $\ell$ of molecule given by

$$\ell n \sigma = 1$$

$n$ = number density $(N/V)$
$\sigma$ = scattering cross-section $\sim \pi r^2$

at STP, $\ell \approx 3\,nm$

Classical approximation of distinguishability is valid for $\ell \gg \lambda$ (de Broglie $\lambda$)

$$\langle p^2 \rangle = 2m \langle E_k^{tr} \rangle = 3mkT$$

$$m_{N_2} = 26\ GeV$$

$$\lambda = \left.\frac{hc}{\sqrt{3mkT}}\right|_{300K} = \frac{1240\ eV\cdot nm}{\left[3\cdot 26\cdot 10^9\, eV\left(\frac{1}{40}eV\right)\right]^{1/2}}$$

$$= 10^{-4}\frac{1240}{\sqrt{19.6}}\,nm = 0.03\,nm = \frac{\ell}{100} \ll \ell$$

**Quantum Statistics** Quantum states labeled by energy $\varepsilon_i$. Generalize Boltzmann probability,

$$P_i = \frac{1}{Z} e^{-\varepsilon_i/kT}$$

$$Z = \sum_{i \text{ all states}} e^{-\varepsilon_i/kT} \equiv e^{-\mu/kT}$$ $\mu(T)$ = chemical potential

Then $P_i = e^{-(\varepsilon_i - \mu)/kT}$ Gibbs factor

In this form, we can generalize to quantum statistics.

Physically $\mu$ is the change of energy when one particle is added to the system (technically, @ constant entropy and volume). Or, $-\mu$ is energy needed to remove one particle.

$\mu(T)$ is weakly dependent on temperature

$\varepsilon - \mu > 0$ always.

<u>Quantum Statistics</u> Each state has probability $P_n$ to be occupied by $n$ particles

$$P_n = \frac{1}{Z} e^{-n(\varepsilon-\mu)/kT}$$

define $x \equiv \frac{\varepsilon-\mu}{kT}$

<u>Fermi-Dirac:</u> $n = 0, 1$

$$P_0 = \frac{1}{Z} \qquad P_1 = \frac{e^{-x}}{Z} \qquad P_0 + P_1 = 1$$

$$Z = 1 + e^{-x}$$

$$\bar{n}_{FD}(\varepsilon) = 0 \cdot P_0 + 1 P_1 = \frac{e^{-x}}{1+e^{-x}} = \frac{1}{e^{x}+1}$$

$$= \left[ e^{(\varepsilon-\mu)/kT} + 1 \right]^{-1}$$

<u>Bose-Einstein:</u> $n = 0, 1, 2, \ldots \infty$

$$P_0 = 1, \quad P_1 = e^{-x}, \quad P_2 = (e^{-x})^2, \quad \ldots$$

$$Z = \sum_{n=0}^{\infty} (e^{-x})^n = \frac{1}{1-e^{-x}}$$ Taylor expand to prove.

$$\bar{n}_{BE}(\varepsilon) = \frac{1}{Z} \sum_{n=0}^{\infty} n e^{-nx} = \frac{1}{Z}\left( \frac{-d}{dx} Z \right)$$

$$\frac{d}{dx} Z = \frac{d}{dx}\left( \frac{1}{1-e^{-x}} \right) = \frac{-e^{-x}}{(1-e^{-x})^2}$$

So

$$\bar{n}_{BE}(\varepsilon) = \left(\frac{e^{-x}}{1-e^{-x}}\right) = \left(\frac{1}{e^{x}-1}\right)$$

$$= \left[e^{(\varepsilon-\mu)/kT} - 1\right]^{-1}$$

notice that $\frac{\varepsilon-\mu}{kT} \gg 1$ gives in both cases the

Classical Maxwell Boltzmann

$$\bar{n}_{MB}(\varepsilon) = e^{-(\varepsilon-\mu)kT}$$

Another derivation (following Rohlf, Eisberg + Resnick)

Equilibrium between two states (a, b) with

number $n_a + n_b$ = constant and energies $\varepsilon_a, \varepsilon_b$

Bosons  Transition rate enhanced by number already in state. Let $W_{ba}$ be rate for transition $b \to a$, $W_{ab}$ rate $a \to b$.

$$W_{ba} = (1+n_a)\, e^{-\varepsilon_a/kT}$$
$$W_{ab} = (1+n_b)\, e^{-\varepsilon_b/kT}$$

In equilibrium at temperature T

$$\frac{n_a}{n_b} = \frac{W_{ba}}{W_{ab}} = \frac{(1+n_a)\, e^{-\varepsilon_a/kT}}{(1+n_b)\, e^{-\varepsilon_b/kT}}$$

rearrange to get

$$\frac{1+n_a}{n_a} e^{-\varepsilon_a/kT} = \frac{1+n_b}{n_b} e^{-\varepsilon_b/kT} \equiv A(T)$$ depends only on T

solve $\left(\frac{1}{n_a} + 1\right) = A e^{\varepsilon_a/kT}$

$$n_a = \frac{1}{A e^{\varepsilon_a/kT} - 1}$$ with $A(T) = e^{-\mu(T)/kT}$

infer

$$\bar{n}_{BE}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

Fermions ; $W_{ba} = (1-n_a)e^{-\varepsilon_a/kT}$ zero if $n_a = 1$

$$\frac{n_a}{n_b} = \frac{(1-n_a)e^{-\varepsilon_a/kT}}{(1-n_b)e^{-\varepsilon_b/kT}}$$

As before, $\frac{1-n_a}{n_a} e^{-\varepsilon_a/kT} \equiv A(T)$

Solve to get $\bar{n}_a(\varepsilon) = \frac{1}{Ae^{\varepsilon_a/kT}+1}$

with $A(T) = e^{-\mu(T)/kT}$ get

$$\bar{n}_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$$

# Quantum Distribution Functions

From Schroeder Thermal Physics

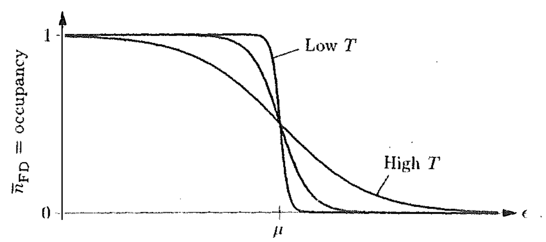



**Figure 7.6.** The Fermi-Dirac distribution goes to 1 for very low-energy states and to zero for very high-energy states. It equals 1/2 for a state with energy $\mu$, falling off suddenly for low $T$ and gradually for high $T$. (Although $\mu$ is fixed on this graph, in the next section we'll see that $\mu$ normally varies with temperature.)

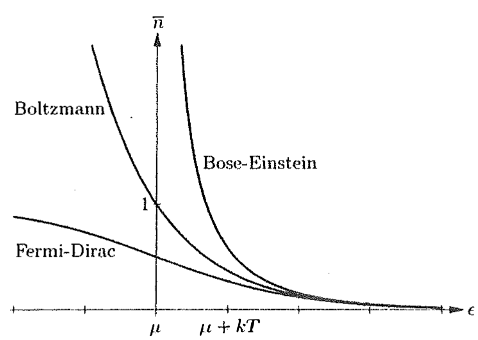



**Figure 7.7.** Comparison of the Fermi-Dirac, Bose-Einstein, and Boltzmann distributions, all for the same value of $\mu$. When $(\epsilon - \mu)/kT \gg 1$, the three distributions become equal.

The parameter $\mu$ is the chemical potential which loosely can thought of as the energy needed to add or remove a particle or molecule from the system. It has dimensions of energy. For the Fermi-Dirac distribution, it is approximately a constant equal to the Fermi energy $\epsilon_F$.

## Electrons in metal

about 1 conducting $e^-$ per atom, giving

$$\left(\frac{V}{N}\right)^{1/3} \sim 0.2\,\text{nm}$$

de Broglie wavelength of $e^-$ @ 300K (room temp.)

$$\lambda = \frac{hc}{\sqrt{3 m_e kT}} = \frac{1240\ \text{eV}\cdot\text{nm}}{\sqrt{3(5\times10^5\,\text{eV})\left(\frac{1}{40}\right)}\ \text{eV}}$$

$$= \frac{1240}{\sqrt{\frac{15}{4}}}\,10^{-2}\,\text{nm} \cong 6\,\text{nm}$$

$e^-$ wavefunctions overlap and we need Quantum statistics.

$$N = \int_0^\infty \rho(\varepsilon)\,\bar{n}_{FD}(\varepsilon)\,d\varepsilon$$

Electron density of states.

Quantized phase space volume is

$$\frac{d^3p\,d^3x}{h^3}$$

non-relativistic $\varepsilon = \frac{p^2}{2m}$

Volume large enough to treat momentum continuous.

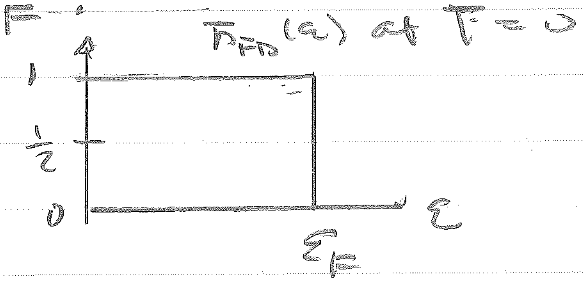
$$\frac{d\varepsilon}{dp} = \frac{p}{m} = \frac{1}{m}\sqrt{2m\varepsilon}$$

$$\rho(\varepsilon)\,d\varepsilon = \underset{\substack{\uparrow \\ \text{2 spin states}}}{2}\,\frac{(4\pi)\,p^2 dp\, V}{h^3}$$

$$= \frac{V\,8\pi(2m\varepsilon)}{h^3}\,\frac{m}{\sqrt{2m\varepsilon}}\,d\varepsilon$$

$$= \frac{V}{h^3}\left(2^{7/2}\pi\right) m^{3/2}\sqrt{\varepsilon}\,d\varepsilon$$

Define Fermi-Energy as $\varepsilon_F = \mu$ at $T=0$.
At $t=0$, all energy states have $\bar{n}=1$ up to $\varepsilon_F$.




$$N = \frac{V}{h^3}\,2^{7/2}\,\pi\, m^{3/2}\int_0^{\varepsilon_F}\sqrt{\varepsilon}\,d\varepsilon = \frac{V}{h^3}\,2^{7/2}\,\pi\, m^{3/2}\,\frac{2\varepsilon_F^{3/2}}{3}$$

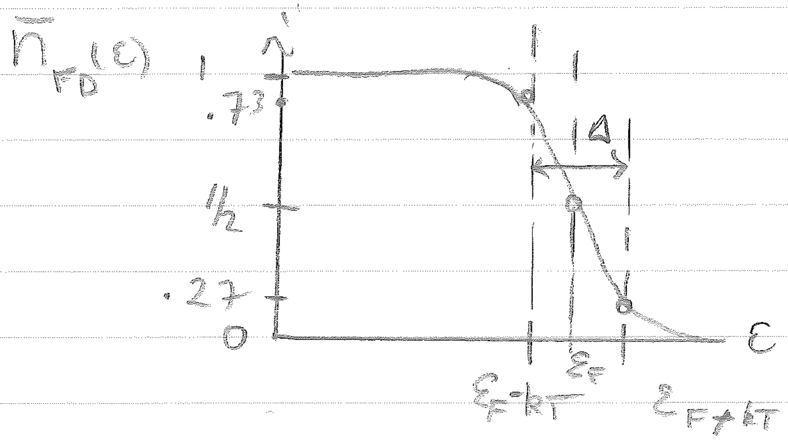
$$\varepsilon_F = \left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}\frac{h^2}{8m}$$

$$8 = \left(2^{9/2}\right)^{2/3}$$

with $\left(\frac{V}{N}\right)^{1/3} = 0.2\,\text{nm}$

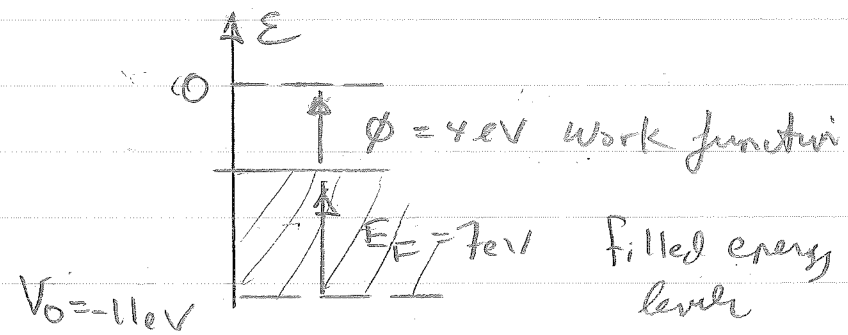
$$\varepsilon_F \cong \left(\frac{1}{0.2\,\text{nm}}\right)^2 \frac{(1240\,\text{eV}\cdot\text{nm})^2}{8\,(5\times10^5\,\text{eV})} = 9.6\,\text{eV}$$

For example $\varepsilon_F$ copper $= 7\,\text{eV}$




At room temperature $\Delta = 2kT \ll \varepsilon_F$

For example,
Copper




$$\Delta = 2\left(\tfrac{1}{40}eV\right) = 0.05\,eV \ll 7eV$$

So we can approximate $\bar{n}_{FD}(\varepsilon) = \begin{cases} 0 & \varepsilon > \varepsilon_F \\ 1 & \varepsilon < \varepsilon_F \end{cases}$

<u>Degeneracy pressure</u>:

total energy $U$ is

$$U = \int_0^\infty \varepsilon\, g(\varepsilon)\, \bar{n}_{FD}(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\, d\varepsilon$$

$$= \left(\frac{V}{h^3}\right) 2^{7/2}\pi\, m^{3/2} \int_0^{\varepsilon_F} \varepsilon^{3/2}\, d\varepsilon$$

$$= \frac{V}{h^3}\, 2^{9/2}\, \frac{\pi}{5}\, m^{3/2}\, \varepsilon_F^{5/2}$$

Expand $\bar{n}_{FD}(\varepsilon)$ near $\varepsilon_F$ with $x = \frac{\varepsilon - \varepsilon_F}{kT} = \frac{\Delta}{2kT}$

$\bar{n}_{FD}(x) = (e^x + 1)^{-1}$ for $x << 1$ $e^x \approx 1 + x$

$$\cong (2 + x)^{-1} = \frac{1}{2}\left(1 + \frac{x}{2}\right)^{-1} \cong \frac{1}{2}\left(1 - \frac{x}{2}\right)$$

$$\bar{n}_{FD}(\Delta) = \frac{1}{2}\left(1 - \frac{\Delta}{4kT}\right)$$

$$\bar{n}_{FD}(0) = 1/2$$

$$\bar{n}_{FD}(\Delta = +2kT) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

$$\bar{n}_{FD}(\Delta = -2kT) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = 3/4$$

rewrite in terms of $\varepsilon_F$,

$$\varepsilon_F = \left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}\frac{h^2}{8m}$$

$$V = \frac{3N}{\pi}\cdot\left(\frac{h^2}{8m}\frac{1}{\varepsilon_F}\right)^{3/2}$$

giving much simplified result

$$U = \frac{3N}{\cancel{\pi}}\ \frac{\cancel{2^{9/2}}}{\cancel{h^3}}\ \frac{\cancel{\pi}}{5}\ \cancel{m^{3/2}}\ \varepsilon_F^{5/2-3/2}\left(\frac{\cancel{h^3}}{\cancel{m^{3/2}}}\right)\frac{1}{\cancel{2^{9/2}}}$$

$$\boxed{U = \frac{3}{5}N\varepsilon_F}$$

Degeneracy pressure:

$$P_D = -\frac{\partial U}{\partial V} = \frac{-3}{5}N\frac{\partial \varepsilon_F}{\partial V} = \frac{3}{5}N\left(\frac{2}{3}\right)\frac{\varepsilon_F}{V}$$

$$= \frac{2}{5}\left(\frac{N}{V}\right)\varepsilon_F$$

Star collapse to white dwarf 

In burnt out star, elecron degeneracy pressure can balance gravitational collapse.

Gravitational self energy is

$$U_G = -G\int_0^R \frac{m\,dm}{r} = -G\frac{(4\pi\rho)^2}{3}\int_0^R r^4 dr = -G\frac{16}{15}(\pi\rho)^2 R^5$$

$m(r) = \rho\left(\frac{4\pi}{3}\right)r^3 \quad dm = 4\pi\rho r^2 dr \quad m\,dm = \frac{(4\pi\rho)^2}{3}r^5 dr$

in terms of total mass $M = \frac{4\pi}{3}\rho R^3$

$$U_G = -G\frac{16\pi^2}{15}\left(\frac{3}{4\pi}\frac{M}{R^3}\right)^2 R^5 = -\frac{3}{5}\frac{GM^2}{R}$$

in terms of $V = \frac{4}{3}\pi R^3$

$$U_G = -\frac{3}{5}\left(\frac{3}{4\pi}\right)^{1/3} G M^2 V^{-1/3}$$

Gravitational pressure

$$P_G = -\frac{\partial U_G}{\partial V} = -\frac{1}{5}\left(\frac{3}{4\pi}\right)^{1/3} G M^2 V^{-4/3}$$

rewrite degeneracy pressure $P_D$ in terms of $V$ and number of nucleons $N_N = 2N_e$

Typical Carbon-Oxygen white dwarf has # neutrons = # protons = $\frac{N_N}{2}$.

$$P_D = \frac{2}{5}\frac{N_e}{V}\varepsilon_F = \frac{2}{5}\frac{N_e}{V}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{N_e}{V}\right)^{2/3}\frac{h^2}{8m_e}$$

$$(N_e)^{5/3} = \left(\frac{N_N}{2}\right)^{5/3} = \left(\frac{M}{m_p}\right)^{5/3}\left(\frac{1}{2}\right)^{5/3} \quad m_p = \text{proton mass}$$

$$P_D = \frac{2}{5}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1}{2}\right)^{5/3}\frac{1}{8}\left(\frac{M}{V}\right)^{5/3}\frac{h^2}{m_e m_p^{5/3}}$$

$$= \frac{1}{5}\left(\frac{3}{\pi}\right)^{2/3}\underbrace{\left(\frac{1}{2}\right)^{5/3}\frac{1}{4}}_{\left(\frac{1}{2}\right)^{11/3}}\left(\frac{M}{V}\right)^{5/3}\frac{h^2}{m_e M_p^{5/3}}$$

@ equilibrium $P_D + P_G = 0$ $P_D = -P_G$

$$\frac{1}{5}\left(\frac{1}{2}\right)^{11/3}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{m}{V}\right)^{5/3}\frac{\hbar^2}{m_e m_p^{5/3}} = \frac{1}{5}\left(\frac{3}{4\pi}\right)^{1/3} G m^2 V^{-4/3}$$

$$R = \left(\frac{3V}{4\pi}\right)^{1/3} = \frac{\hbar^2}{G m_e m_p^{5/3}} M^{-1/3} \underbrace{\left(\frac{1}{2}\right)^{11/3}\left(\frac{3}{\pi}\right)^{2/3}}_{C \equiv 0.076}$$

(I think this constant is correct!)

$$\frac{1}{m_p^{5/3}} m^{-1/3} = \frac{1}{m_p^2}\left(\frac{m_p}{m}\right)^{1/3}$$

$$\frac{\hbar^2}{G m_p^2 m_e} = \frac{(6.6\times10^{-34}\ \text{J}\cdot\text{s})^2}{(6.67\times10^{-11}\ \text{m}^3/\text{kg}/\text{s}^2)(1.67\times10^{-27}\ \text{kg})^2(9\times10^{-51}\ \text{kg})}$$

$$= 0.26\times10^{28}\ \text{m} = 2.6\times10^{24}\ \text{km}$$

In terms of solar mass $\frac{M_\odot}{m_p} = \frac{2\times10^{30}\ \text{kg}}{1.67\times10^{-27}\ \text{kg}} = 1.2\times10^{57}$

$$R = C \times \frac{2.6\times10^{24}\ \text{km}}{(1.2\times10^{57})^{1/3}}\left(\frac{M_\odot}{m}\right)^{1/3} = C\ 2.44\times10^{5}\ \text{km}\left(\frac{M_\odot}{m}\right)^{1/3}$$

in terms of $R_\odot = 7\times10^5$ km

$$R = C\left(\frac{2.44}{7}\right) R_\odot \left(\frac{M_\odot}{m}\right)^{1/3} = 0.026\, R_\odot \left(\frac{M_\odot}{m}\right)^{1/3}$$

Sun will collapse to white dwarf

for $m > 1.44 M_\odot$, degeneracy pressure is not enough to prevent neutron star or black hole

## Photon gas at temperature T.

Photon number is not conserved so $\mu = 0$

$$\bar{n}_\gamma(\varepsilon) = \left(e^{\varepsilon/kT} - 1\right)^{-1}$$

$\varepsilon = pc$ degeneracy = 2 polarizations

$$U = \frac{2V}{h^3}\int \underbrace{d^3p}_{4\pi p^2 dp}\ \bar{n}_\gamma(\varepsilon)\,\varepsilon$$

$$= \frac{2V\,4\pi}{(ch)^3}\int_0^\infty \frac{\varepsilon^3\,d\varepsilon}{e^{\varepsilon/kT}-1}$$

$$= \frac{8\pi V}{(ch)^3}(kT)^4 \underbrace{\int_0^\infty \frac{x^3\,dx}{e^x - 1}}_{\pi^4/15}$$

$$\frac{U}{V} = \frac{8\pi^5}{15}\frac{k^4}{(ch)^3}T^4 \equiv \int_0^\infty u(\varepsilon)\,d\varepsilon$$

and

$$u(\varepsilon) = 8\pi\left(\frac{kT}{hc}\right)^3\left[\frac{\left(\frac{\varepsilon}{kT}\right)^3}{e^{\varepsilon/kT}-1}\right]$$

Black body spectrum from lecture #4.

## Bose-Einstein condensate

BE statistics circa 1905

London (1938) superfluidity, superconductivity

1995 BEC in Lab Nobel 2001

rubidium @ $170 \times 10^{-9}$ K — Cornell, Weisman

Sodium — Ketterle

{ laser-cooled magnetically trapped gas }

$$\bar{n}_{BE}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

$$N = \sum_{\text{states}} \bar{n}_{BE}(\varepsilon)$$

in limit $T \to 0$, all atoms are in ground state $\varepsilon = \varepsilon_0$. Then

$$N = \lim_{T \to 0} \frac{1}{e^{(\varepsilon_0-\mu)/kT} - 1}$$

implies $\mu < \varepsilon_0$. Since $N$ is large ($\sim 10^{23}$)

$$e^{(\varepsilon_0-\mu)/kT} \sim 1 \quad \text{or} \quad \frac{\varepsilon_0-\mu}{kT} \ll 1 .$$

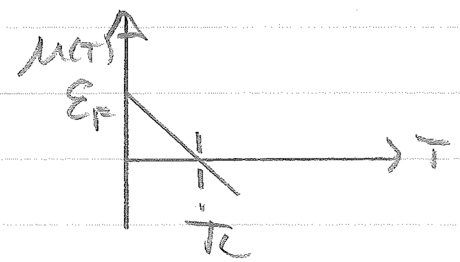
Taylor expand

$$e^{\varepsilon_0-\mu/kT} - 1 \approx \frac{\varepsilon_0-\mu}{kT} \quad \text{then}$$

$$N^{-1} = \frac{\varepsilon_0-\mu}{kT} \quad \text{near } T=0 \text{ and}$$

$$\mu(T) \cong \varepsilon_0 - \frac{kT}{N} \quad \text{indeed, } < \varepsilon_0$$

Some atoms reach critical temperature where $\mu(T_c)=0$




Well above $T_c$, number of atoms condensed to ground state $n_0 \ll N$. Then ignore condensate and $\varepsilon$ is essentially continuous.
Confining atoms in cubic box of volume V.

$$N=(2j+1)\frac{V}{h^3}\int_0^\infty 4\pi p^2\,dp\left[\frac{1}{e^{(\varepsilon-\mu)/kT}-1}\right]$$

$\varepsilon = p^2/2m \quad d\varepsilon = \frac{p}{m}dp = \frac{\sqrt{2m\varepsilon}}{m}dp$ we get

$$N=(2j+1)\frac{V}{h^3}\int_0^\infty 4\pi m\sqrt{2m\varepsilon}\;\bar{n}_{BE}(\varepsilon)\,d\varepsilon$$

$$=(2j+1)V\,4\pi\sqrt{2}\left(\frac{mkT}{h^2}\right)^{3/2}\int_0^\infty\frac{\sqrt{x}\,dx}{e^{(x-\mu/kT)}-1}$$

At $T_c$, $\mu=0$ we can do the integral

$$\int_0^\infty\frac{\sqrt{x}\,dx}{e^x-1}=\frac{\sqrt{\pi}}{2}\zeta(3/2) \qquad \zeta(3/2)=2.612$$

Solve for $T_c$:

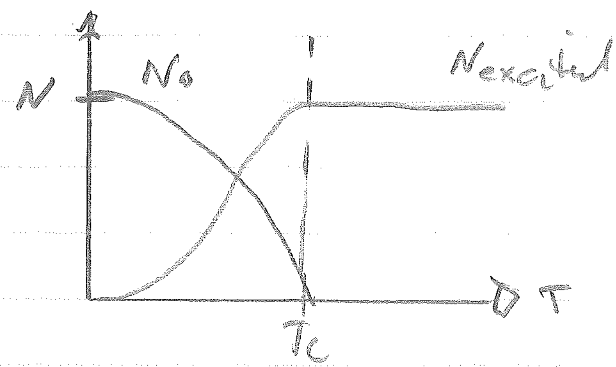
$$N=(2j+1)V\left(\frac{2\pi mkT_c}{h^2}\right)^{3/2}\zeta(3/2)$$

$$T_c = \frac{h^2}{2\pi m k}\left(\frac{N}{(2j-1)\zeta(3/2)}\right)^{2/3}$$

Below $T_c$, Einstein predicted some fraction of atoms collapse to the ground state.

$$N = N_0 + N_{excited} = N_0 + N\left(\frac{T}{T_c}\right)^{3/2}$$

$$N_0 = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]$$

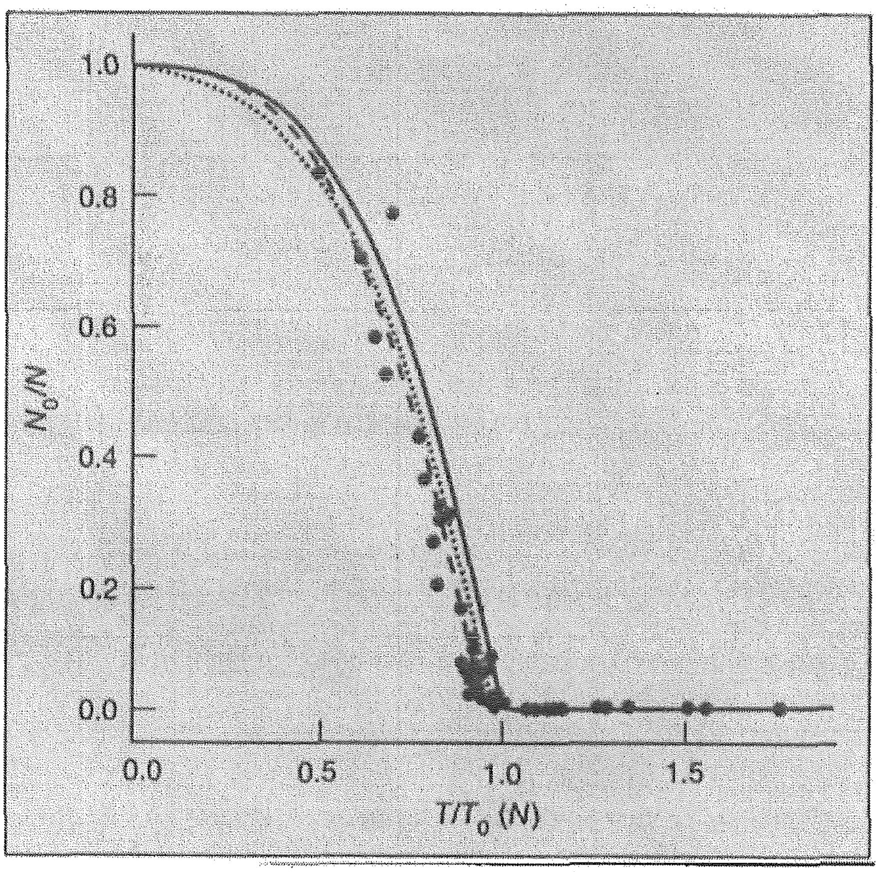



For harmonic trap, $N_0 = N\left[1 - \left(\frac{T}{T_c}\right)^{3}\right]$

16-15

**Date:** February 3, 2020 at 3:29 PM

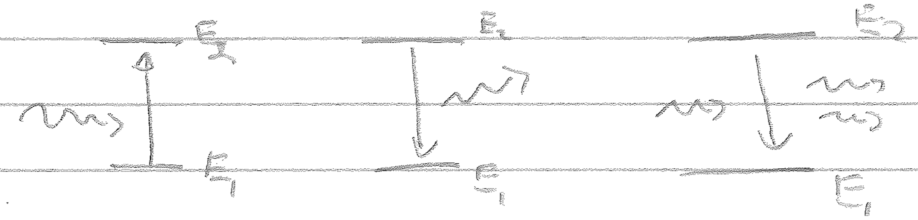
From **Bose-Einstein condensation** CHRISTOPHER TOWNSEND, WOLFGANG KETTERLE AND SANDRO STRINGARI
Physics World, March 1997




**5** Condensate fraction, $N_0/N$, measured as a function of scaled temperature, $T/T_0$, at Boulder. $T_0$ is the predicted critical temperature for an ideal gas in a harmonic trap in the thermodynamic limit (many atoms). The solid curve is the predicted dependence in the thermodynamic limit, and the dotted curve includes a correction for the finite number of atoms (4000) in the condensate. To the uncertainties in the data, the measurement is consistent with the theory for non-interacting bosons. The dashed line is a best fit to the experimental data.

Lasers two energy states separated by $E_2 - E_1 = hf \equiv \varepsilon$

Einstein Coefficents.



absorption | spontaneous emission | stimulate emission
B | A | B'

$P_A = A \cdot \Delta t$

$P_B = B\,u(\varepsilon)\,\Delta t$ $\quad u(\varepsilon)$ photon energy density per unit volume U/V

$P_{B'} = B'\,u(\varepsilon)\,\Delta t$

$$\frac{dN_1}{dt} = N_2 A - N_1 B u(\varepsilon) + N_2 B' u(\varepsilon)$$

$$\frac{dN_2}{dt} = -N_2 A + N_1 B u(\varepsilon) - N_2 B' u(\varepsilon)$$

at equilibrium $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$
at temp. T

and $\frac{N_2}{N_1} = e^{-\varepsilon/kT}$

energy density U/V

recall $u(\varepsilon) = 8\pi \left(\frac{kT}{hc}\right)^3 \left[\frac{\left(\frac{\varepsilon}{kT}\right)^3}{e^{\varepsilon/kT} - 1}\right]$

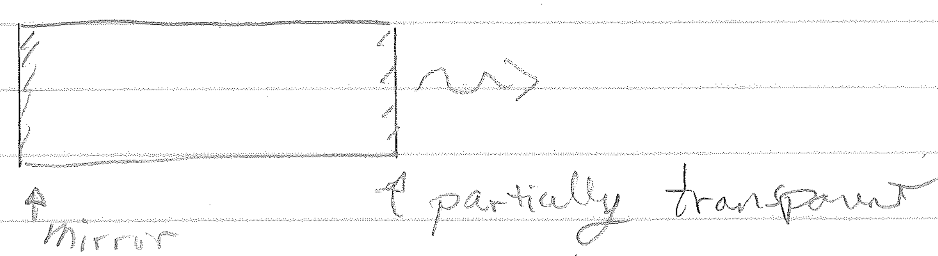
$\frac{dN_1}{dt} = 0$ gives

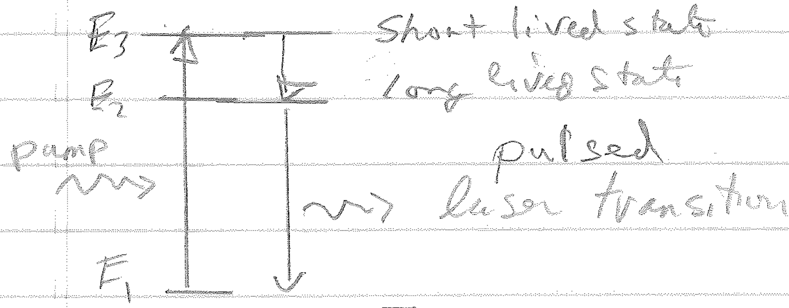
$$\frac{A}{u(\varepsilon)} = B e^{\varepsilon/kT} - B'$$

$$\frac{A}{8\pi\left(\frac{kT}{hc}\right)^3 \left(\frac{\varepsilon}{kT}\right)^3} \left[e^{\varepsilon/kT} - 1\right] = B e^{\varepsilon/kT} - B'$$

$$\boxed{\text{so } B = B' \text{ and } A = 8\pi\left(\frac{kT}{hc}\right)^3 \left(\frac{\varepsilon}{kT}\right)^3}$$

To get lasing, need a population inversion $N_2 > N_1$




Townes 1954