Phys 330 Spring 2021

Lecture #17 : Scattering

Ordinary "baryonic" matter composed of atoms.
Atoms have structure - internal excited states.
Probe structure by

a) observe atomic state decays

b) scattering

Through scattering, atom → nucleus → nucleons (proton, neutron)
→ quarks

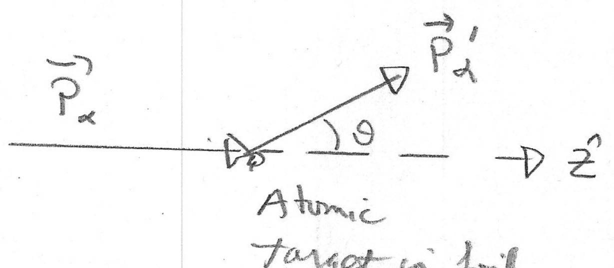
Quarks are structureless, point-like particles.

Scattering experiments are of two types -
fixed target, colliding beams.

Distance probed is related to momentum through
de Broglie relation (holds relativistically)

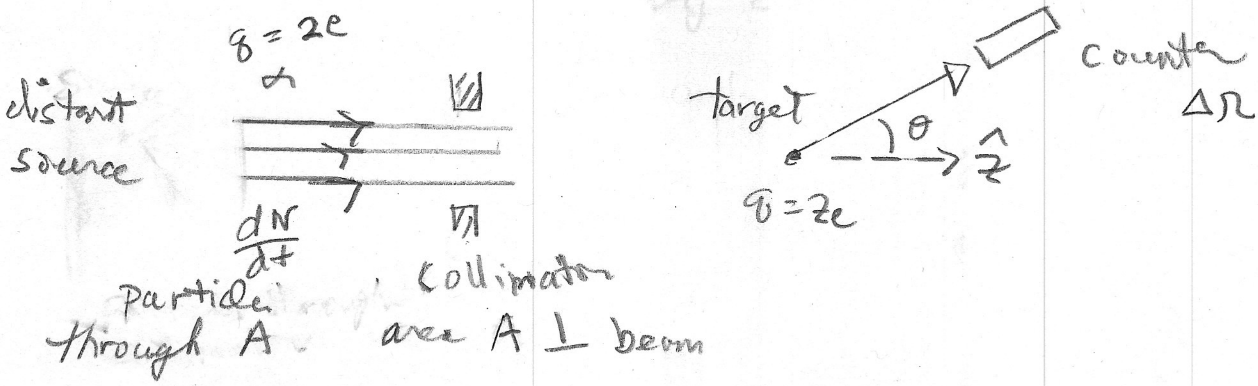
$p = h/\lambda$ short distance → high energy

Rutherford experiment : α (He^{++}) on gold foil $Z = 79$, $A = 197$



α For atom with $Z=79$ electrons and mass $A=197$ assumed to be uniformly electrically neutral and size $\sim a_0$, expect only small angle scattering.

Rutherford et al. observed suprising large angle scattering.



Beam flux $F = \left(\frac{dN}{dt}\right)\frac{1}{A}$

(flux)(area)=dN/dt

Counter solid angle $\Delta\Omega$ centered at θ, ϕ.

differential counting rate (ϕ independent) $\frac{dR}{d\Omega}(\theta)$

normalized scattering rate is

$$\Delta\sigma = \frac{\left(\frac{dR}{d\Omega}\right)\cdot\Delta\Omega}{F} \qquad \frac{\text{Counting rate}}{\text{Beam Flux}}$$

differential cross section

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{1}{F}\frac{dR}{d\Omega}(\theta)$$

Physics is independent of beam flux

Total cross section

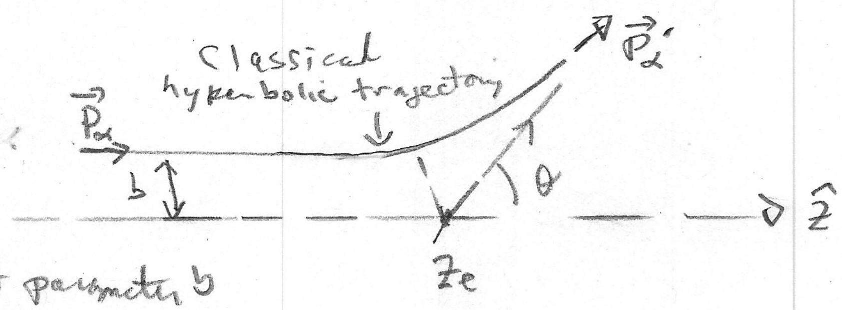
$$\sigma = \int_{4\pi} \frac{d\sigma}{d\Omega}\, d\Omega = \frac{\text{total rate } (1/\text{time})}{\text{Flux } (1/\text{time Area})}$$

dimension = area

area $\perp$ beam, σ Lorentz invariant for boosts along beam direction.

Classical, non-relativistic scattering theory:

$m_\alpha / m_{AU} = 4/197 \ll 1$ fixed target (no recoil)



impact parameter b

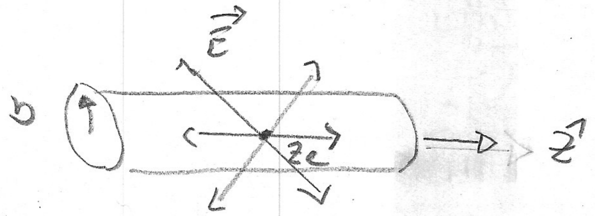
Change in momentum $\perp$ to beam

$$\Delta P_\perp = \int_{\text{trajectory}} F_\perp\, dt = 2e \int E_\perp\, dt = 2e \int \frac{E_\perp}{v}\, dz$$

$v \cong$ constant along trajectory for small angle

$$\Delta P_\perp = \frac{2e}{v} \int E_\perp\, dz$$

use Gauss's law over cylinder centered on z of radius impact parameter b.



$$\vec{E} = \frac{kze}{r}\hat{r}$$

$$e^2 k = \alpha \hbar c$$

$$\oint_{\text{cylinder } b} \vec{E}\cdot d\vec{a} = 4\pi k ze$$

$$2\pi b \int_{-\infty}^{\infty} E_\perp \, dz = 4\pi k (ze)$$

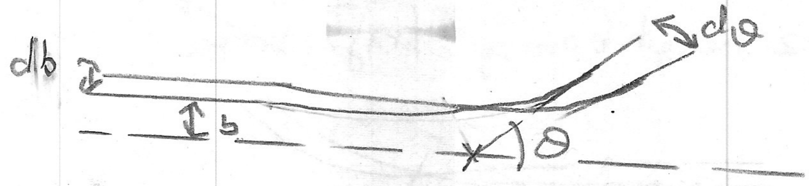
then $$\Delta P_\perp = \left(\frac{2e}{v}\right)\left(\frac{4\pi k ze}{2\pi b}\right) = \frac{4z\alpha\hbar c}{vb}$$

scattering angle is

$$\theta(b) = \frac{\Delta P_\perp}{P_\parallel} = \left(\frac{4z\alpha\hbar c}{vb}\right)\frac{1}{m_\alpha v}$$

$$\boxed{\theta(b) = \frac{2\alpha z\hbar c}{b E_\alpha}}$$

Consider two neighboring trajectories:



all particle in annular region between b, b+db get scattered into angle between θ, θ+dθ. Then

$$F\,(2\pi\, b\, db) = \underbrace{\frac{dR}{d\Omega}}_{\frac{d\sigma}{d\Omega}\cdot F}\cdot (2\pi \sin\theta\, d\theta)$$

flux cancels:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

Classical non-relativistic differential cross section

Rutherford cross section: $\theta(b) = \frac{2\alpha Z \hbar c}{E_\alpha}\left(\frac{1}{b}\right) \Rightarrow$

$$\left|\frac{db}{d\theta}\right|^{-1} = \left|\frac{d\theta}{db}\right| = \frac{2Z\alpha\hbar c}{E_\alpha}\frac{1}{b^2} \qquad b = \frac{2\alpha Z\hbar c}{E_\alpha}\frac{1}{\theta}$$

$$\left|\frac{db}{d\theta}\right| = b^2\left(\frac{E_\alpha}{2Z\alpha\hbar c}\right) = \frac{2\alpha Z\hbar c}{E_\alpha}\frac{1}{\theta^2}$$

For small angle $\sin\theta \approx \theta$; $\frac{b}{\theta} = \frac{2\alpha Z\hbar c}{E_\alpha}\frac{1}{\theta^2}$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\alpha Z\hbar c}{E_\alpha}\right)^2 \frac{1}{\theta^4}$$

Rutherford small angle

Exact classical result:

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4}\left(\frac{\hbar c}{E_\alpha}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

<u>Remarks:</u>

(1) exact quantum result is the same!

(2) diverges as $\theta \to 0$ $(b \to \infty)$

Coulomb force of nucleus screened by atomic electrons, so divergence is artifact of calculation.

$\theta_{min} = 10^\circ = 0.17$ radians for 10 MeV α

$$b_{max} = \frac{2Z\alpha\hbar c}{E_\alpha}\frac{1}{\theta_{min}} = \frac{2(79)}{137}\left(\frac{200\,\text{MeV fm}}{10\,\text{MeV}}\right)\left(\frac{1}{0.17}\right) = 118\text{ fm}$$

$$= 10^{-3}\,\text{Å} \ll \text{atomic size}$$

We cannot calculate <u>nuclear size</u> from Rutherford σ which assumes point-like nucleus. Distance probed for a head-on scattering @ 10 MeV

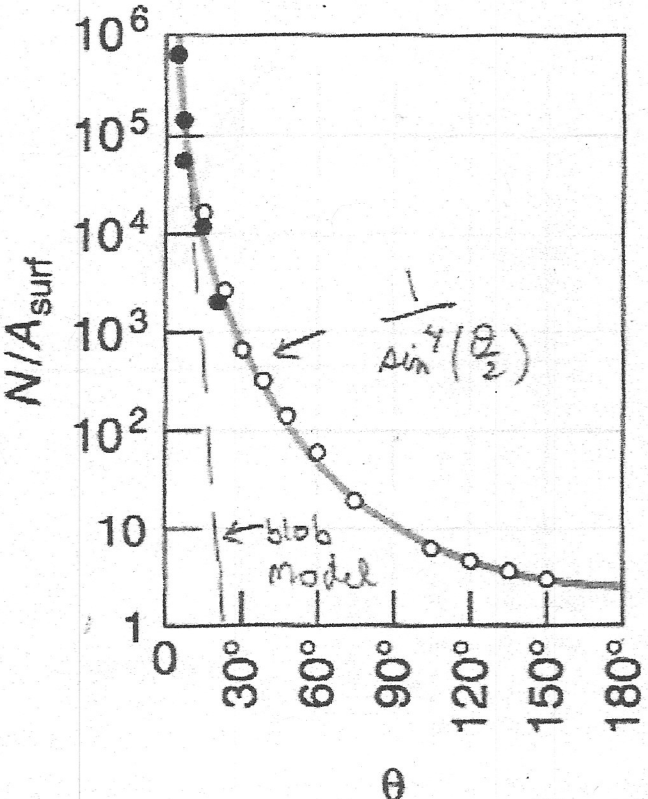
$$E_\alpha = \left(\frac{1}{r_0}\right) 2Z\alpha\hbar c$$

$$r_0 = \frac{2(97)}{137}\,\frac{200\,\text{MeV fm}}{10} = 20\text{ fm}$$

Gold nuclear size $r_{AU} \doteq (1.2\,fm)(179)^{1/3}$ (hard-sphere packing of nucleons)

$= 6.8\ fm$

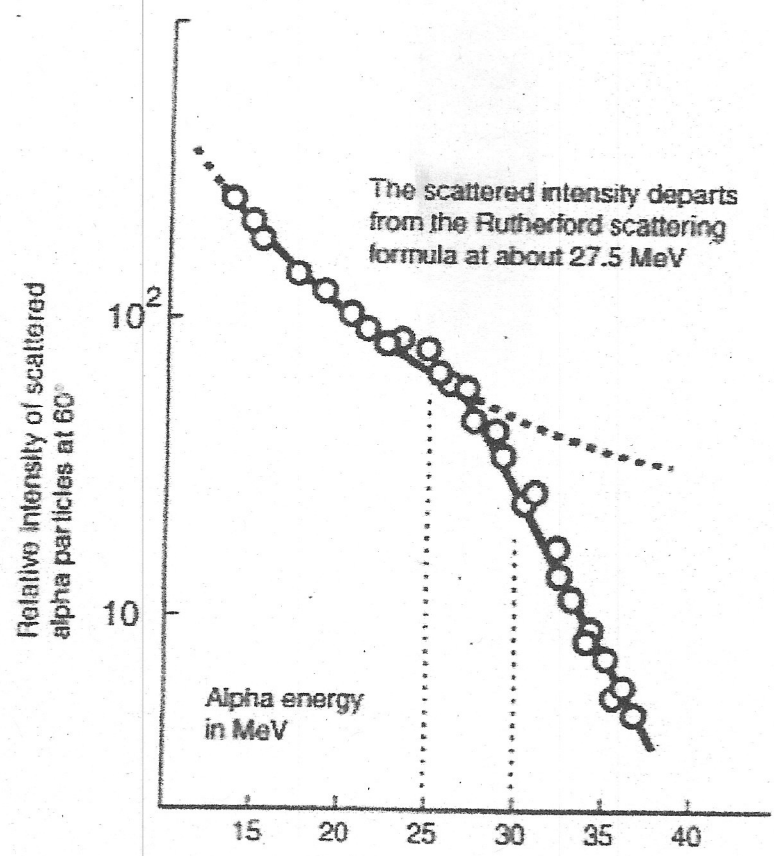
blob model = mass distributed according to atomic scale.
large angle scattering was astonishing to Rutherford



Geiger, Marsden data $E_\alpha = 17.7\,MeV$

Follows point-like scattering of Rutherford σ

Seeing the nucleus -



Finite charge size modifies σ by square of "form factor". Take simple charge density

$$\rho(r) = \rho_0 e^{-r/r_0}$$

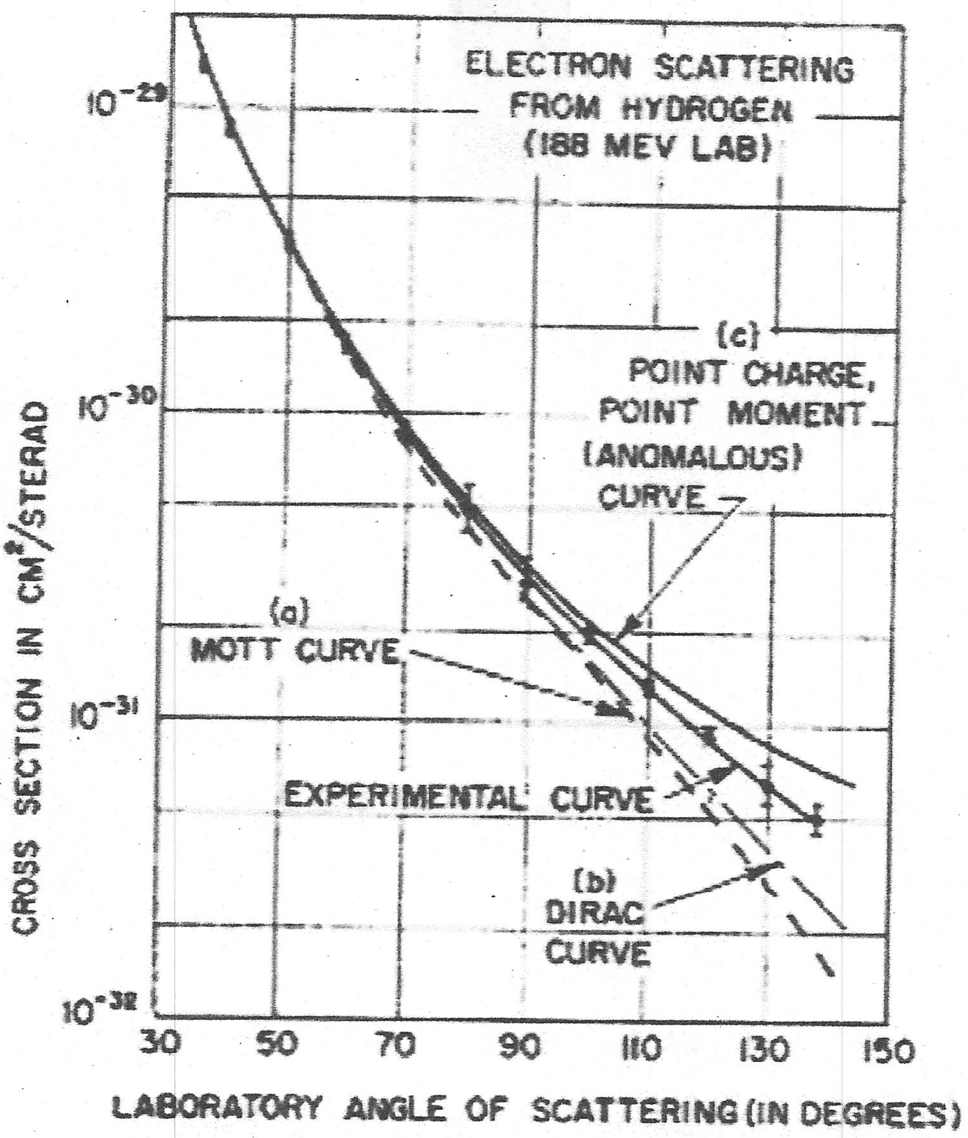
Form factor is Fourier transform of ρ

$$F(q) = \left(1 + \frac{q r_0}{\hbar c}\right)^{-1}$$ $\vec{q}$ is momentum transfer

$$|\vec{q}|^2 = |\vec{p} - \vec{p}\,'|^2 \underset{\text{elastic}}{=} 2p^2(1-\cos\theta) = 4E_\alpha m_\alpha(1-\cos\theta)$$

finite size of proton

PRL 102, 851 (1956)



fitted $r_0 = 0.24$ fm