Phys 330 Spring 2020

Lec #3 : Energy - Momentum

Momentum $\vec{F} = \frac{d\vec{p}}{dt}$

For isolated system, $\vec{P}$ is conserved.

For example in collisions,

$$\sum \vec{P}_i^{\,initial} = \sum \vec{P}_i^{\,final}$$

In a deep way, momentum conservation is a consequence of translational invariance of laws of physics.

At non-relativistic speeds $\vec{p} = m\vec{v}$

Example : illustrating CM

Elastic collision

Equal mass

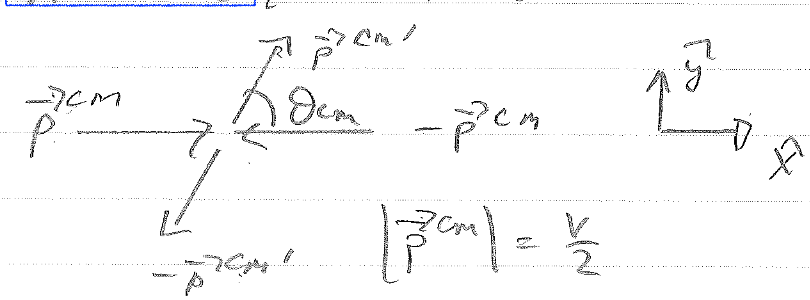
$\vec{p}_1 = mv\hat{x}$ $\quad$ m $\quad$ $\vec{P}_2 = 0$ $\quad$ $\hat{x}$

Transform to inertial frame with total (3) momentum $= 0$. (CM frame)

Galilean : $v' = V - V_r$

$$P'_{TOT} = 0 = [m(v - V_r) + m(-V_r)] = 0$$

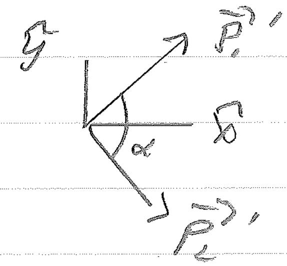
so $\boxed{V_r = V/2}$, In cm frame



$|\vec{p}^{\,cm}| = \frac{V}{2}$

In lab frame after collision

$$\vec{V} = \vec{V}' + \frac{V}{2}\hat{x}$$



$$\vec{P}_1' = m\left(\frac{v}{2}\cos\theta_{cm} + \frac{v}{2}\right)\hat{x} + m\frac{v}{2}\sin\theta_{cm}\,\hat{y}$$

$$\vec{P}_2' = m\left(-\frac{v}{2}\cos\theta_{cm} + \frac{v}{2}\right)\hat{x} - m\frac{v}{2}\sin\theta_{cm}\,\hat{y}$$

$$\vec{P}_1' \cdot \vec{P}_2' = m\left(\frac{v}{2}\right)^2\left[(\cos\theta+1)(-\cos\theta+1) - \sin^2\theta\right]$$

$$= m\left(\frac{v}{2}\right)^2\left[1 - \cos^2\theta - \sin^2\theta\right] = 0$$

so $\alpha = \frac{\pi}{2}$

in special case of elastic collision of equal mass particle.

$\boxed{\text{4-momentum}}$

Construct Lorentz invariant 4-vector analogous to N.R. $\vec{p}$: so

$$\Delta\tilde{x} = \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix}$$

remember, c multiplies time
but I use units where c=1

invariant $(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2$ $\quad m\dfrac{\Delta\tilde{x}}{\Delta\tau}$ also a 4-vector

$$\frac{\Delta t}{\Delta\tau} = \frac{\Delta t}{\sqrt{(\Delta t)^2-(\Delta x)^2}} = \frac{1}{\sqrt{1-\left(\frac{\Delta x}{\Delta t}\right)^2}} \xrightarrow[\lim_{\Delta t\to 0}]{} \frac{1}{\sqrt{1-v^2}} = \gamma$$

$$\lim_{\Delta t\to 0} \frac{dt}{d\tau} = \gamma$$

sometimes I write E=p_0
0 component of 4-vector

$$\tilde{p} = m\gamma\frac{d\tilde{x}}{dt} = m\gamma\begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} E \\ p_x \end{pmatrix}$$

since for boost along $\hat{x}$, $\Delta y' = \Delta y$, $\Delta z' = \Delta z$

complete 4-vector is

$$\tilde{p} = m\gamma\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} \equiv \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$$

Invariant $\tilde{p}\cdot\tilde{p} = m^2 = E^2 - |\vec{p}|^2$

putting back c for clarity: all components have dimension of energy

$$c\tilde{p} = m\gamma c\begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$

$$c\tilde{p} = \begin{pmatrix} E \\ c\vec{p} \end{pmatrix} = \begin{pmatrix} m\gamma c^2 \\ m\gamma \vec{v} \end{pmatrix}$$

note $\gamma = \dfrac{E}{mc^2}$ convenient way to get gama

Non relativistic limit –

$$\gamma = \cosh\theta = \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$$

$$E \cong mc^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right) = \underbrace{mc^2}_{\text{rest}} + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic}}$$

$$\vec{P} = m\vec{v}\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right) \cong m\vec{v}$$

We almost always measure ΔE, so mc^2 term is almost always irrelevant. We recover N.R. KE, $\vec{p}$. In classical physics, mass is constant.

Note: "$m\gamma$" is not consistently interpreted as "moving mass"

$$\vec{F} = m\frac{d}{dt}(\gamma\vec{v}) \neq m\gamma\frac{d\vec{v}}{dt}$$

"moving mass" x acceleration is wrong

Note: $\boxed{\text{Convenient to measure } E \text{ in eV.}}$

Exp. Charged particle kinetic energy comes from accelerating voltage.

Relativistic KE

KE $\quad E_k = \int \vec{F} \cdot d\vec{x} = \int \frac{d\vec{p}}{dt} \cdot d\vec{x}$

$$= \int \frac{d\vec{p}}{dt} \cdot \vec{v} \, dt$$

$$\frac{d}{dt}\left(E^2 - p^2\right) = 0$$

$$\frac{d}{dt}\left(m^2 \gamma^2 - p^2\right) = 0$$

$$m^2 \, 2\gamma \frac{d\gamma}{dt} = 2p \frac{dp}{dt}$$

with $\quad p = m\gamma v \qquad \frac{p}{\gamma} = mv$

$$m \frac{d\gamma}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{v}$$

$$E_K = m \int \frac{d\gamma}{dt} dt = m \int d\gamma = m(\gamma_F - \gamma_I)$$

starting from $v = 0$, $\gamma_I = 1$ and

$$\boxed{E_K = m(\gamma - 1)}$$

Equivalence of mass & energy

$$E = \gamma m$$

nuclear binding energy is large enough to be measureable as mass deficit

$$M(\underbrace{He^{++}}_{\alpha\text{ particle}}) < 2m_p + 2m_n$$

Binding energy

$$E_b = 2[m_p + m_N] - m_\alpha$$
$$= 2[938.27 + 939.57] - 3727.41 \text{ MeV}$$
$$= 28.3 \text{ MeV}$$

have to keep 5 significant figures!

Atomic binding energy is not measureable as mass deficit

$$M(H) = m_p + m_e - E_b \qquad E_b = 13.6\text{ eV}$$
$$= 10^9\text{ eV} + 0.511\times10^6\text{ eV} - 13.6\text{ eV} \cong m_p$$

more about mass deficit when we study nuclear physics.

Creating matter from kinetic Energy

Relativistic Quantum mechanics predicts anti-particle – equal mass, opposite electric charge. Dirac, 1928

Particle can be created, annihilated in particle - antiparticle pairs.

example

particle	anti particle
e^-	e^+ (positron)
p	$\bar{p}$
n	$\bar{n}$ (anti neutron)
photon γ	γ (same as anti-particle)

get meaning of γ from context

example. $p + p \rightarrow p + p + p + \bar{p}$

KE threshold for p beam on p target.

In lab frame

$$\tilde{P}_i = \begin{pmatrix} E+m \\ \vec{p} \end{pmatrix}^{LAB} \quad \text{where } E^2 - p^2 = m^2$$

In CM frame, $\tilde{P}_f = \begin{pmatrix} 4m \\ 0 \end{pmatrix}^{CM}$

since $\tilde{P}_i = \tilde{P}_f$

$$\tilde{P}_i \cdot \tilde{P}_i = \tilde{P}_f \cdot \tilde{P}_f = 16m^2$$

$$16\,m^2 = (E+m)^2 - p^2 = E^2 - p^2 + 2mE + m^2$$
$$= 2mE + 2m^2$$

$$E = 7m \quad \& \quad \gamma = \frac{E}{m} = 7$$

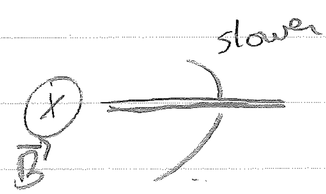
KE $E_k = (\gamma - 1)m = 6m = 6\text{ GeV}.$

$\bar{p}$ discovered at Berkeley Bevatron, 1955
Segrè & Chamberlain.

Positronium bound state of e^+e^- forms
and then $e^+e^- \to \gamma\gamma$.
Lifetime of positronium $\approx 10^{-10}$ s

$E_\gamma = m_e = 511$ keV
Photon spectrum from center of
milky way shows characteristic
positronium annihilation line.

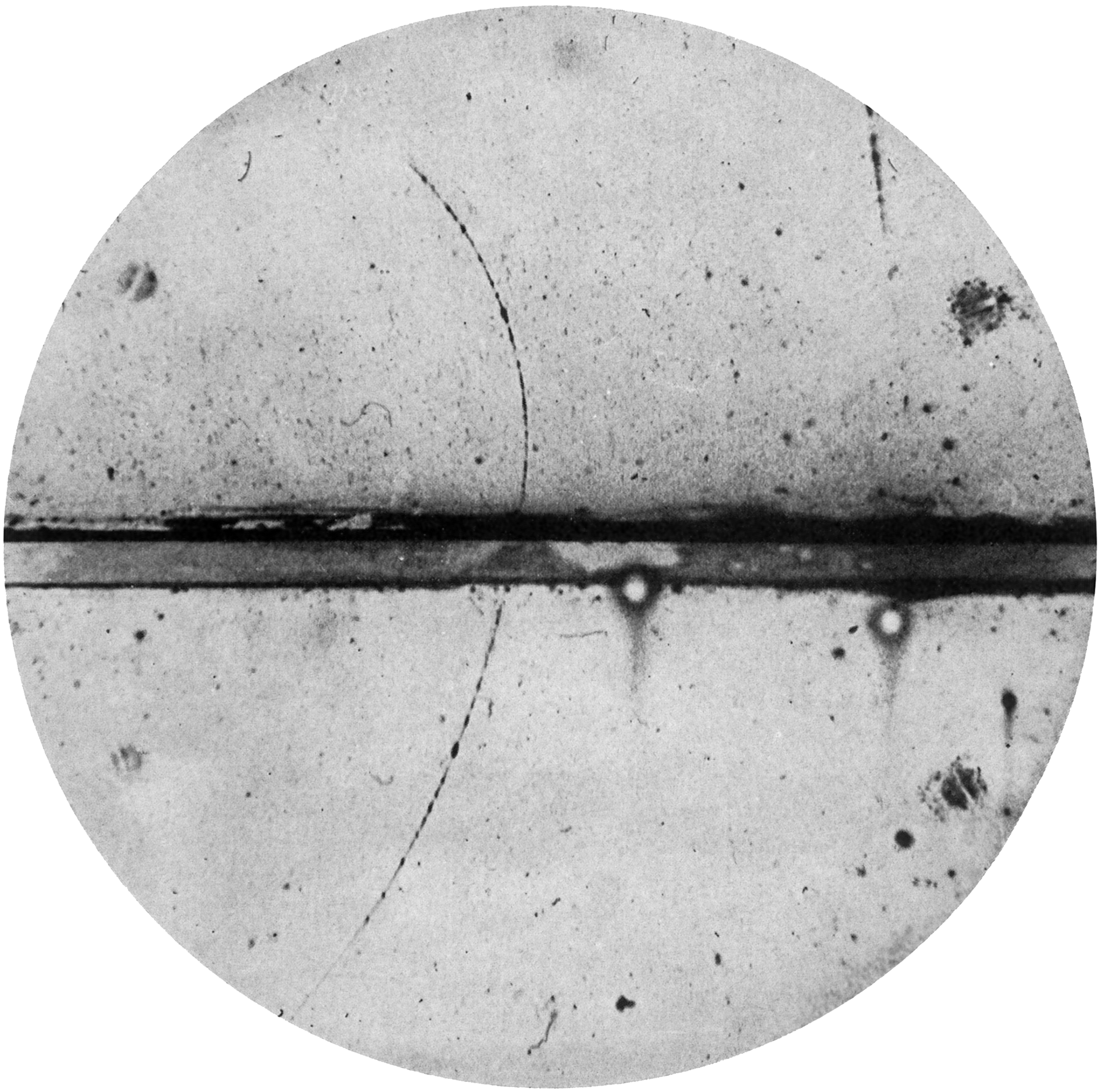
Positron discovered in cosmic rays
1932, Anderson
Cloud chamber and lead plate in
magnetic field.



$\vec{F} = q\vec{v} \times \vec{B}$
$q > 0$
$m \sim m_e$ by $\frac{dE}{dx}$ in pb

water vapor condenses on ionization track uniform magnetic field with direction into page

without lead plate, electron from top would give same curve as positron from bottom.



<u>Massless particle</u> $\tilde{p} = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix} \underset{c=1}{=} \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

$E^2 - \vec{p}\cdot\vec{p} = m^2$

γ	v/c
1	0
2	0.866
3	0.993
...	...
10	0.995

for $\gamma \gg 1$, $v = (1-\gamma^{-2})^{1/2} \doteq 1 - \frac{1}{2}\left(\frac{1}{\gamma^2}\right)$

and $E \approx p$

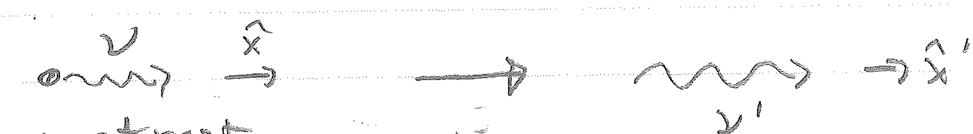
relativistic massless particle

suppose $E^2 - p^2 = 0$ exactly $\rightarrow \frac{v}{c} = 1$

massless

So boost to particle rest frame by $\gamma \rightarrow \infty$ does not exist. Particle travels with $\beta = 1$ (speed c) in all inertial frames.

Relativistic Doppler



source at rest in frame S

observed in frame moving in $+\hat{x}$ direction with speed β

$$E = h\nu$$

$$E' = h\nu' = \gamma(E - v\underbrace{p}_{=E}) = \gamma(1-v)h\nu$$

$$\boxed{\nu' = \frac{1-v}{\sqrt{1-v^2}}\,\nu = \sqrt{\frac{1-v}{1+v}}\,\nu}$$