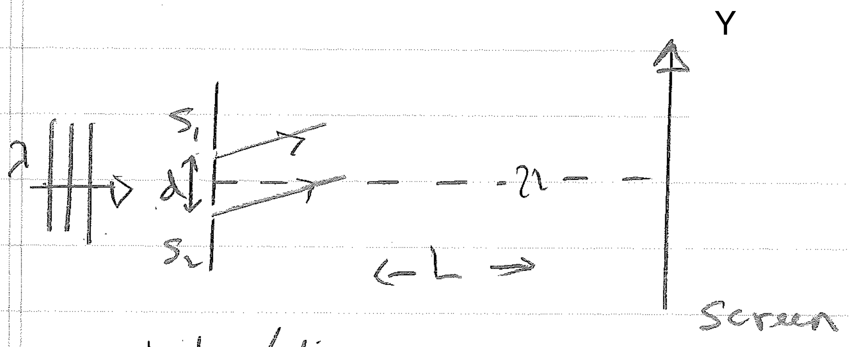
Lec #-5 : Uncertainty Principle

duality - Q.M. object propagates as wave
interacts as particle

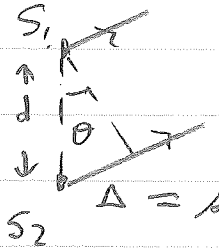
Wave propagation - interference



2 holes/slits
d apart

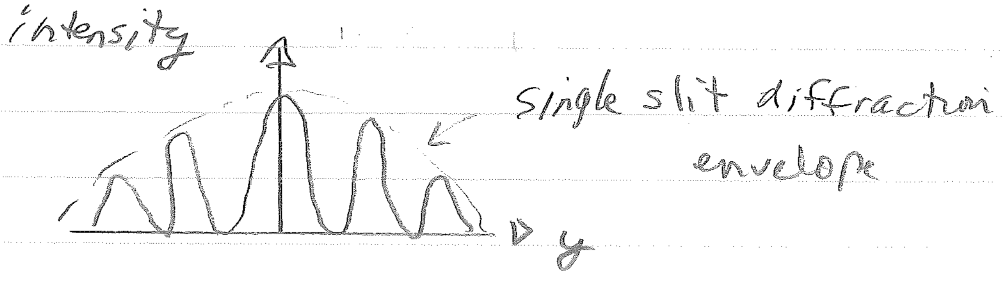
distance $L >> d$

holes S_1, S_2 act like point sources of spherical waves that are in phase.
Interference results from path difference



$$\Delta = \sin\theta \, d \approx \theta d \cong \frac{y}{L} d$$

$$\Delta = \frac{y}{L} d = \begin{cases} n\lambda & \text{constructive} \\ (n+\frac{1}{2})\lambda & \text{destructive} \end{cases}$$



Wave amplitude for plane wave

$$A(x,t) = A_0 \cos(\underbrace{kx - \omega t}_{\phi \equiv \text{phase}})$$

$\omega = 2\pi f$; $k = \frac{2\pi}{\lambda}$

phase velocity $d\phi = k\,dx - \omega\,dt = 0$

$$V_{phase} = \frac{dx}{dt} = \frac{\omega}{k} = f\lambda$$

Interference is result of superposition

$$A_1 + A_2 = A_0 \cos(kx - \omega t) + A_0 \cos\Big(k(x+\Delta x) - \omega t\Big)$$

convenient to use complex amplitudes and take the real part in the end.

In Q.M., complex amplitudes are <u>required</u>.

Complex numbers

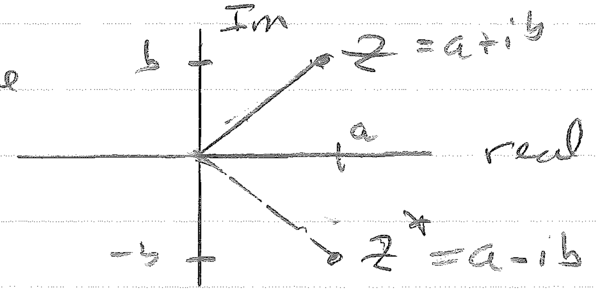
$x^2+1=0$ solution $\sqrt{-1} \equiv i$

a complex number $z = \underset{\substack{\uparrow \\ \text{real part}}}{a} + i\,\underset{\nwarrow \text{ imaginary part}}{b}$

$z_1 + z_2 = (a_1+a_2) + i\,(b_1+b_2)$

$$z_1 z_2 = (a_1+ib_1)(a_2+ib_2) = \\ a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$$

complex plane



complex conjugate

$|z| = \sqrt{a^2+b^2} = \sqrt{zz^*}$

Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{\pi i} = -1$

$z = |z|e^{i\phi}$ $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

and $\cos\theta = \frac{e^{i\phi}+e^{-i\phi}}{2}$ $\sin\theta = \frac{e^{i\phi}-e^{-i\phi}}{2i}$

So $A_1+A_2 = A_0 e^{i(kx-\omega t)}\left(1+e^{ik\Delta x}\right)$

$$= A_0 e^{i(kx-\omega t)} e^{ik\frac{\Delta x}{2}}\left(e^{ik\frac{\Delta x}{2}}+e^{-ik\frac{\Delta x}{2}}\right)$$

$$I = \left|A_1+A_2\right|^2 = 4A_0^2 \cos^2\left(\frac{k\Delta x}{2}\right)$$

$$\frac{k\Delta x}{2} = \frac{2\pi}{\lambda}\frac{\Delta x}{2} = \pi\left(\frac{\Delta x}{\lambda}\right)$$

$$\Delta x = \begin{cases} n\pi & \text{constructive} \\ \left(\frac{2n+1}{2}\right)\pi & \text{destructive} \end{cases}$$

Born interpretation:

Q.M. complex amplitude $\Psi(x,t)$
"wave function"

$|\Psi|^2\Delta x$ is *probability* for e^-/γ to arrive at position between x, $x+\Delta x$

particles detected one at a time

diffraction pattern emerges *statistically*

Wave behavior follows from Heisenberg uncertainty relation

$$\Delta x \Delta p_x \gtrsim \frac{\hbar}{2}$$

Size of atom

Classical hydrogen atom would have size of proton $r \sim 1\,fm$ ($10^{-15}m$)

Bohr radius $a_0 = 0.053\,nm = 0.53 \times 10^5\,fm$

We can estimate Q.M. size from uncertainty principle: binding of e^- to proton is a measure of its location,

$$r \cdot p \sim \hbar$$

take $\sim$ to be $=$ (hindsight!)

$$E = \frac{p^2}{2m} - \frac{ke^2}{r} \qquad e^- \text{ non-relativistic}$$

$k = \frac{1}{4\pi\epsilon_0}$ with e in Coulombs. But for any unit of electric charge,

$$ke^2 = \alpha \hbar c \quad \text{where } \alpha \cong \frac{1}{137} \quad \text{fine structure constant}$$

minimize energy

$$E(r) = \frac{1}{2m}\left(\frac{\hbar}{r}\right)^2 - \frac{\alpha \hbar c}{r}$$

$$\left.\frac{dE}{dr}\right|_{r_{min}} = \frac{2\hbar^2}{2m}\left(-\frac{1}{r^3}\right) + \frac{\alpha \hbar c}{r^2}$$

$$r_{min} = \frac{\hbar^2}{m}\left(\frac{1}{\alpha \hbar c}\right) = \frac{\hbar c}{\alpha m c^2} = 0.053$$

Bohr radius

$$\boxed{\text{Confinement of particle implies minimizing } \langle E_k \rangle \text{ through the uncertainty principle}}$$

Electron inside the proton

$$(\Delta p)c = \frac{\hbar c}{2\Delta x} = \frac{200 \text{ meV} \cdot \text{fm}}{2(1 \text{ fm})} = 100 \text{ meV}$$

$$\gg m_e c^2 = \frac{1}{2} \text{ meV}$$

So

$$E_K = E - m_e c^2 \approx pc$$

$$\langle E_K \rangle = 100 \text{ mev}$$

maximum energy of e^- in free neutron decay $\tau_N = 882$ s $\approx$ 15 minutes free neutron decay

$$n \to p + e^- + \bar{\nu}_e$$

$$E_K^{max} = 1 \text{ mev}$$

Same arguement of $\bar{\nu}_e$. $e^-, \bar{\nu}_e$ are not neutron but created in decay process.

Time Energy Uncertainty

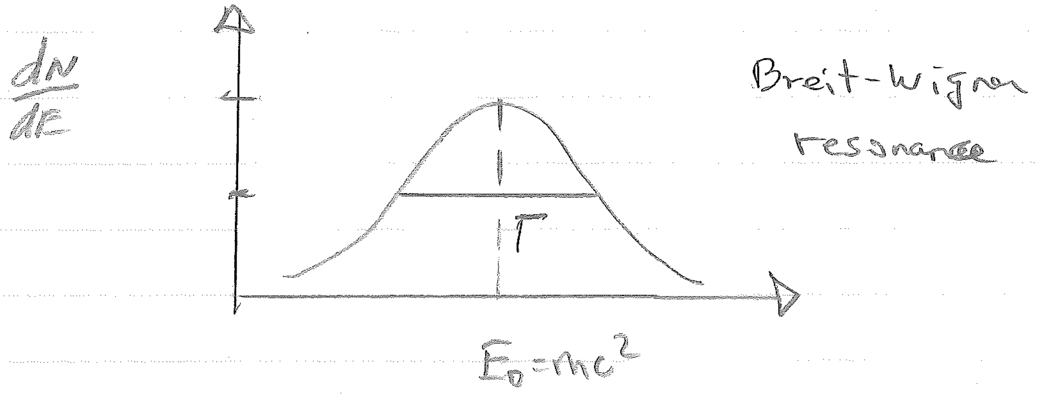
$$\Delta E \cdot \Delta T \geq \frac{\hbar}{2}$$

A particle of finite lifetime τ has an uncertainty in it's mass of

$$\Delta E = \frac{\hbar}{2\tau}$$

Measured "line width" $\Gamma = 2\Delta E = \frac{\hbar}{\tau}$

For very short lived particle, cannot measure τ directly but infer from width



Γ is full width at half maximum

for neutron, $\tau \cong 10^3 s$

$$\frac{\Delta m_N}{m_N} = \frac{\Gamma}{mc^2} = \left(\frac{\hbar c}{c\tau}\right)\frac{1}{mc^2}$$

$$= \frac{200 \text{ MeV}\cdot\text{fm}}{(3\times10^8 \text{m/s})(10^3 s)(1000 \text{MeV})} \doteq 10^{2-15-8} = 10^{-21}$$

So neutron has a definite mass.

The charged, weak boson W has

$$\frac{\Gamma_W}{m_W c^2} = \frac{2\text{GeV}}{80.4\text{GeV}} = 2.5\%$$

lifetime of W

$$\tau_W = \frac{1}{c}\left(\frac{\hbar c}{\Gamma}\right) = \frac{1}{c}\left(\frac{200 \text{ MeV}\cdot\text{fm}}{2\text{ MeV}}\right)$$

$$= \frac{10^{-13}\text{m}}{3\times10^8 \text{m/s}} = \frac{1}{3}10^{-21} s$$