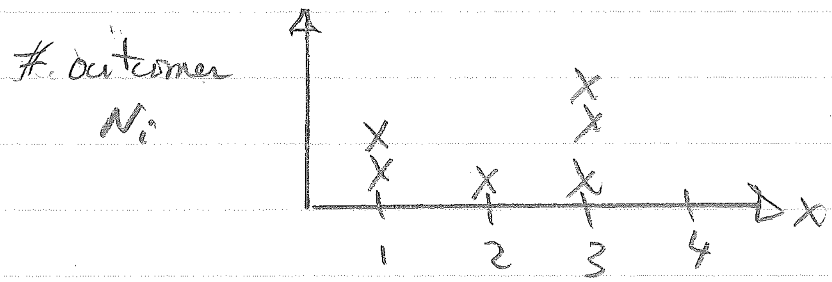
Lec #6: Elements of Probability

Born interpretation, Q.M. complex amplitude Ψ has $|\Psi|^2$ probability density

$$|\Psi|^2 \Delta x = \text{prob. to find particle between } x,\ x+\Delta x$$

So we need to review probability ideas.

Start with quantity x that can take on descrete values. Measure x N times with outcomes $\{x_1, x_2, \ldots x_N\}$. Make a frequency plot (histogram)



Estimate probability to get value x_i,

$$P(x_i) \simeq \frac{\#\,\text{trials} = x_i}{N_{\text{trials}}} = \frac{N_i}{N}$$

here $P(3) \simeq \frac{3}{6} = \frac{1}{2}$

$$\sum_{i=1}^{N} P(x_i) = 1$$

probabilities always sum to one.

Average or mean or expectation value:

$$\langle X \rangle = \bar{X} = \frac{\sum x_i N_i}{\sum N_i} = \sum X_i P_i$$

$$P_i \equiv P(x_i)$$

$$\langle X^2 \rangle = \sum x_i^2 P_i$$

Variance $V \equiv (\Delta X)^2 = \langle (X - \langle x \rangle)^2 \rangle$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

ΔX is measure of statistical uncertainty

For x taking <u>continuous</u> values:

$$P_i \longrightarrow P(x)\, dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, P(x)\, dx$$

P(x) is a probability density function (PDF)

Example - exponential decay

$$\frac{dN}{dt} = -\frac{N(t)}{\tau}$$

$$N(t) = N(0)\, e^{-t/\tau}$$

normalized PDF is $P(t) = \frac{1}{\tau} e^{-t/\tau}$

$$\int_{0}^{\infty} P(t)\, dt = 1$$

$$\langle t \rangle = \frac{1}{\tau} \int_0^\infty t\, e^{-t/\tau}\, dt = \tau \int_0^\infty x e^{-x}\, dx = \tau$$

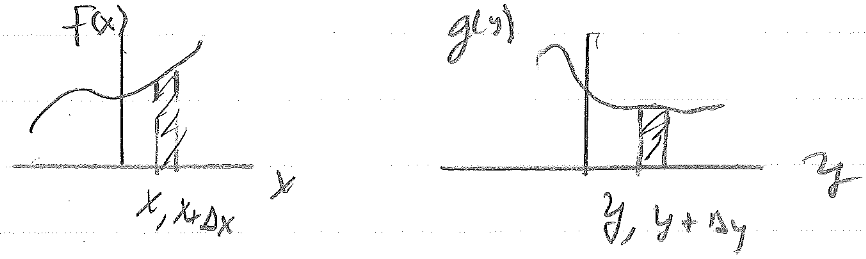
useful integral: $\int_0^\infty x^n e^{-x}\, dx = n!$

$$V = (\Delta t)^2 = \frac{1}{\tau} \int_0^\infty e^{-t/\tau} (t-\tau)^2\, dt$$

$$= \tau^2 \int_0^\infty e^{-x} (x-1)^2\, dx = \tau^2 \left[2! - 2 + 1 \right] = \tau^2$$

$$\boxed{\Delta t = \sqrt{V} = \tau}$$

Change of variables probability is conserved



$$f(x)\,dx = g(y)\,dy$$ where $y = h(x)$ is change of variables

$$g(y)\,dy = \left|\frac{dh}{dx}\right|^{-1} dy\, f(x(y))$$

$$y = h(x)$$
$$x = h^{-1}(y)$$

$$g(y) = \left|\frac{dh}{dx}\right|^{-1} f(x(y))$$

$$f_\lambda(\lambda) = hc \frac{8\pi}{\lambda^5} \left(\frac{1}{e^{hc/\lambda kT} - 1} \right)$$

Central limit theorem -

Given n statistically independent (uncorrelated) random variables r_i with finite mean, variance

$$y = \frac{\sum_{i=1}^{N} r_i}{N} \quad \text{tends toward a Gaussian}$$

In limit $n \to \infty$, y is Gaussian

let means of r_i be μ_i, variance $(\Delta r_i)^2$ as $N \to \infty$

$$\langle y \rangle = \lim_{N\to\infty} \frac{1}{N} \sum \mu_i$$

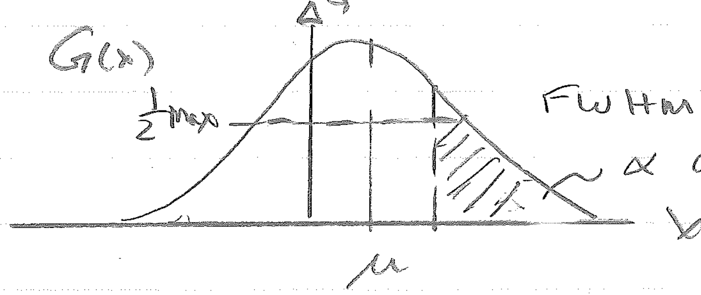
$$(\Delta y)^2 = \lim_{N\to\infty} \frac{1}{N} \sum (\Delta r_i)^2$$

The Gaussian :

$$G(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right\}$$

$$\int_{-\infty}^{\infty} G\, dx = 1 \qquad \langle x \rangle_G = \mu \qquad (\Delta x)_G = \sigma$$

σ parameter is Gaussian error. Often we say "$V = \sigma^2$"



$\text{FWHM} = 2\sigma\sqrt{2\ln 2} = 2.35\sigma$

α area under positive tail

Note that log G is parabolic

probability content

α	δ
31.73%	σ
4.55%	2σ
0.27%	3σ
6.3×10^{-5}	4σ

α	δ
20%	1.28σ
10%	1.64σ
5%	1.96σ
1%	2.58σ

Uncorrelated random variables

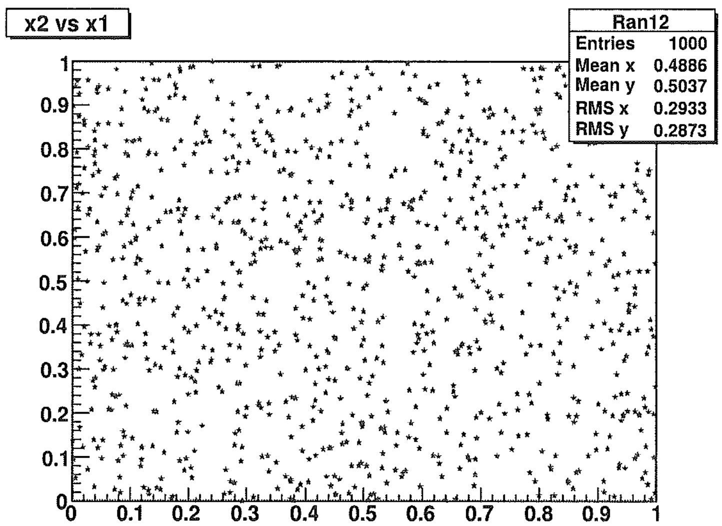


Figure: two uncorrelated, “flat” random variables

Average of N (uncorrelated) random variables:

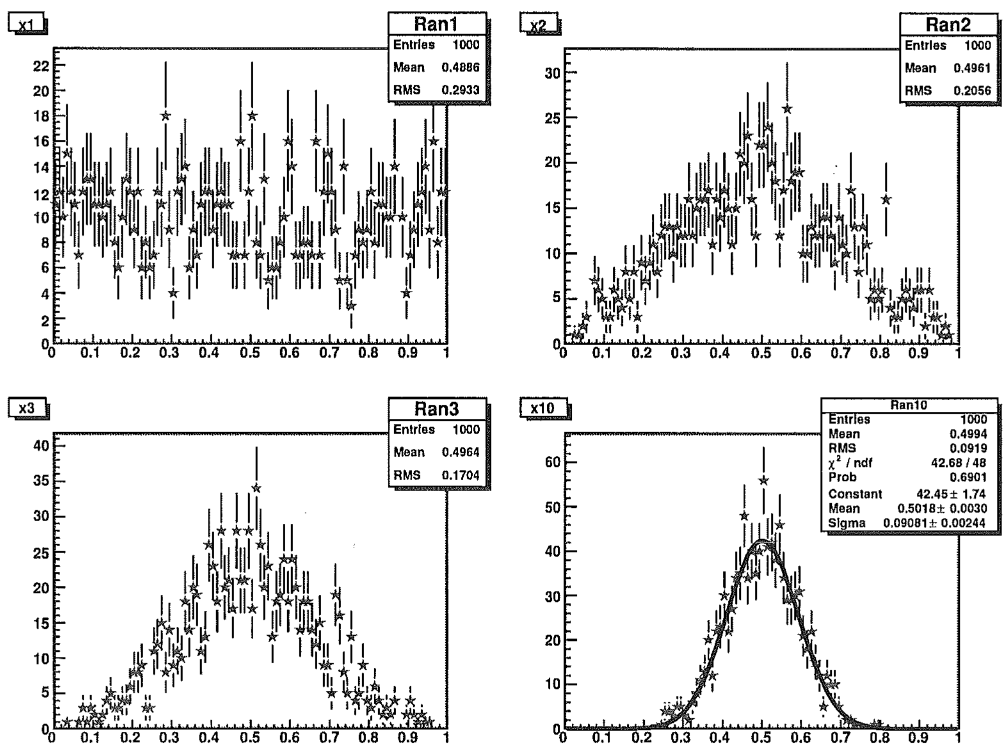


Figure: bottom right (average of 10) is fitted to a Gaussian