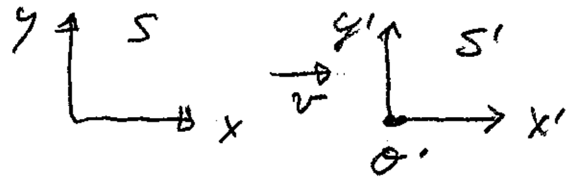
Phys 450

Spring 2021

HW #1 Solutions

① A boost leaves $\bar{x}\cdot\bar{x} = t^2 - x^2$ invariant



$$\bar{O}' \xrightarrow[\text{frame } S]{} (t, vt) \quad , \quad \bar{O}' \xrightarrow[\text{frame } S']{} (t', 0)$$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$x' = 0 = t\left(B_{21} + B_{22} v\right) \quad ; \quad v = -B_{21}/B_{22}$$

$t'^2 - x'^2 = t^2 - x^2$ leads to 3 equations

1. $B_{11}^2 - B_{12}^2 = 1$

2. $B_{11} B_{12} - B_{21} B_{22} = 0$

3. $B_{22}^2 - B_{12}^2 = 1 = B_{22}^2 (1 - v^2) = 1$

$$B_{22} = 1/\sqrt{1-v^2} \equiv \gamma$$

2. gives $B_{11} B_{12} = -v\gamma^2$

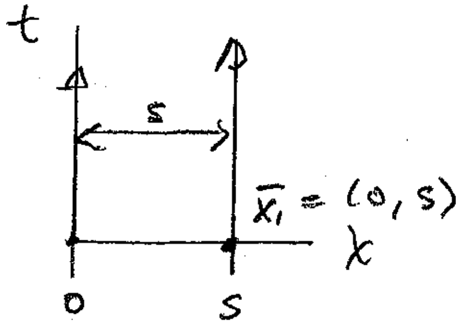
Substitution in 1. gives $B_{11}^2 - \dfrac{v^2\gamma^4}{B_{11}^2} = 1 \Rightarrow B_{11} = \gamma$

so the boost is $\begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix}$

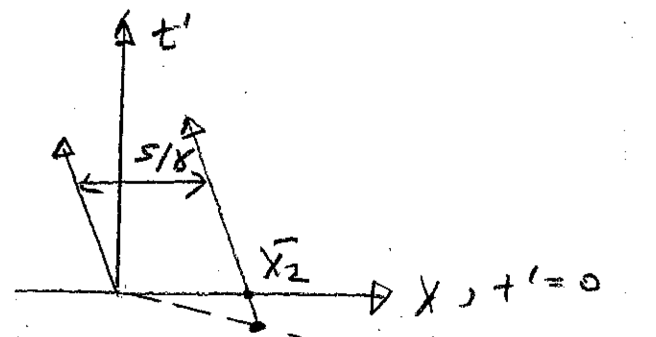
② proper length

Define frames S, S' as in problem ①

The object is of length s at rest in the unprimed frame. A sketch of the world lines of the Left and Right ends in frame S,



In the primed frame,



line of t=0

$\bar{X}_1 = (t_1', x_1')$

$$\begin{pmatrix} t_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} = \begin{pmatrix} -v\gamma s \\ \gamma s \end{pmatrix}$$

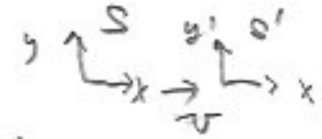
at $t'=0$, the right end of the object is at

$$x_2' = \gamma s - v(v\gamma s) = \gamma s(1-v^2) = s/\gamma$$

So length of moving object, measured at $t'=0$ is s/γ. Length of moving object is contracted.

#3

$$\tanh\theta = v$$



use tilde for 4-vectors

$$\bar{B}(\theta) = \begin{pmatrix} \cosh\theta, & -\sinh\theta \\ -\sinh\theta, & \cosh\theta \end{pmatrix}$$

when $\tilde{x} = (t, x)$
just x space component

$$\bar{B}(\theta_1)\cdot\bar{B}(\theta_2) = \begin{pmatrix} c_1 & -s_1 \\ -s_1, & c_1 \end{pmatrix}\begin{pmatrix} c_2 & -s_2 \\ -s_2, & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1c_2 + s_1s_2, & -c_1s_2 - s_1c_2 \\ -s_1c_2 - c_1s_2, & s_1s_2 + c_1c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\theta_1+\theta_2), & -\sinh(\theta_1+\theta_2) \\ -\sinh(\theta_1+\theta_2), & \cosh(\theta_1+\theta_2) \end{pmatrix} = B(\theta_1+\theta_2)$$

hyperbolic trig identities

Addition of velocities $\tanh\theta_{12} = v_{12}$

$$v_{12} = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1\tanh\theta_2} = \frac{v_1 + v_2}{1 + v_1v_2}$$

#4 (3.3)

(a) Volume $V = \Delta x \cdot \Delta y \cdot \Delta z$

under L.T. in x direction, $V' = \frac{\Delta x}{\gamma} \cdot \Delta y \, \Delta z = \frac{V}{\gamma}$

(b) density $\rho = \frac{N}{V}$ $\quad$ N is Lorentz invariant

so $\rho' = \gamma \rho$

#5 (3.4) (a) $d_{NR} = v \Delta t$

(b) In earth frame $\Delta t = \gamma \Delta\tau$ (τ proper time)

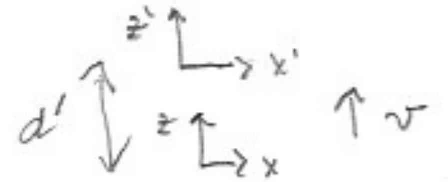
distance $d = v \Delta t = v \gamma \Delta\tau = \gamma d_{NR}$

$\gamma(v = 0.998) = 15.8$

$d_{NR} = (2.2 \times 10^{-6} \text{sec})(3 \times 10^{8} \text{m/s}) = 660 \text{ms}$

$d = 10.3$ km

(c) in muon rest frame (primed)



$d' = \Delta\tau \, v$

$= \frac{d}{\gamma}$

distance traveled is Lorentz contracted.

(d)

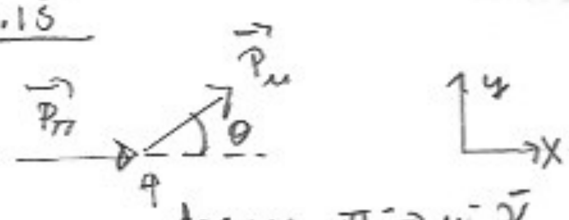
Charged Pion lifetime π^+ is $\tau_{\pi^+} = 2.6 \times 10^{-8}$s

$d_{\pi^+} = \gamma \Delta\tau \, c = 122$ m $= 0.1$ km

charged pions do not make it to ground,

but neutrinos do.

#6 3.15

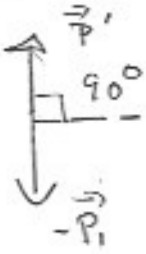


Lab frame

decay $\pi^- \to \mu^- \bar{\nu}$

In rest frame of pion (zero 3 momentum frame)

m_π decay



Write four vector w/o z component as

$\tilde{p} = (E, p_x, p_y)$ then in pion rest frame

4-momentum conservation is

$$(m_\pi, 0, 0) = (E_\mu', 0, p') + (p', 0, -p') \quad (m_\nu = 0)$$

$$m_\pi = E_\mu' + p' = \sqrt{m_\mu^2 + p'^2} + p'$$

$$p' = \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \equiv \frac{m_\pi}{2}(1 - r^2) \qquad r = \frac{m_\mu}{m_\pi}$$

Transform to lab frame

$$\begin{pmatrix} E \\ p_x \\ p_y \end{pmatrix}_\mu = \begin{pmatrix} \gamma & \gamma v & 0 \\ \gamma v & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_\mu' \\ 0 \\ p' \end{pmatrix}$$

$$E_\mu' = \sqrt{m_\mu^2 + p'^2}$$

$$\tan\theta = \frac{p_y^u}{p_{xu}} = \frac{p'}{\gamma v E_u'} \xrightarrow[m_u \to 0]{} \frac{1}{\gamma v}$$

this is called the headlight effect.

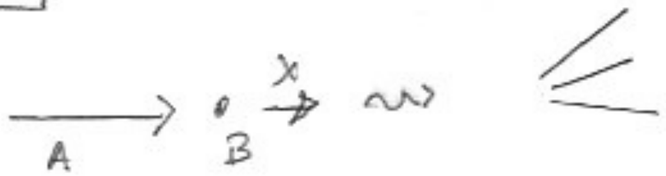
$$\left(\frac{E_u'}{p'}\right)^2 = \frac{m_u^2 + p'^2}{p'^2} = \frac{m_\pi^2\left(r^2 + m_\pi^2 p'^2\right)}{p'^2}$$

$$= m_\pi^2\left[\frac{r + \frac{1}{4}(1-r^2)}{\frac{m_\pi^2}{4}(1-r^2)^2}\right]$$

$$= \frac{4r^2 + 1 - r^2}{(1-r^2)^2} = \frac{3r^2+1}{(1-r^2)^2}$$

$$\tan\theta = \left(\frac{1}{\gamma v}\right)\frac{1-r^2}{\sqrt{3r^2+1}}$$

$\boxed{\#7\ ,\ 3,16}$



C_i final
N final state
Particle

In Zero 3 momentum frame

$$P_f' = (M, \vec{0}) \qquad M = \sum_{i=1}^{N} m_i \quad \text{sum of masses}$$

with 4-momentum dropping irrelevant y, z components

$$\tilde{P}_{TOT} = (E_A, P_A) + (m_B, 0)$$

$$P'_{TOT} = (M, 0)$$

4-vector dot product is L.I.

$$\tilde{P}_{TOT} \cdot \tilde{P}_{TOT} = (E_A + m_B)^2 - P_A^2 = M^2$$

$$E_A = \sqrt{m_A^2 + P_A^2}$$

$$E_A^2 + 2m_B E_A + m_B^2 - P_A^2 = M^2$$

$$2m_B E_A = M^2 - m_A^2 - m_B^2$$

$$E_A = \frac{1}{2m_B}\left[M^2 - m_A^2 - m_B^2\right]$$